Gradient Magnetometry with Atomic Ensembles

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Abstract

We study gradient magnetometry with ensembles of atoms with arbitrary spin. We calculate precision bounds for estimating the gradient of the magnetic field based on the quantum Fisher information (qFI). For states that are sensitive to homogeneous fields, a simultaneous measurement is needed, as the homogeneous field must also be estimated.

We present a method to calculate precision bounds for gradient estimation with two spatially separated atomic ensembles. We also consider a single atomic ensemble with an arbitrary density profile, where the atoms cannot be addressed individually, and which is a very relevant case for experiments.

System setup

- The system $\varrho = \varrho^{(x)} \otimes \varrho^{(s)}$ spans along the x-axis

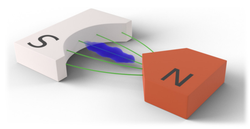
$$\varrho^{(x)} = \int \frac{P(\boldsymbol{x})}{\langle \boldsymbol{x}|\boldsymbol{x}\rangle} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \, \mathrm{d}\boldsymbol{x}$$

- The magnetic field is linear in x

$$\boldsymbol{B} = \boldsymbol{B}_0 + x\boldsymbol{B}_1 + \mathcal{O}(x^2)$$

- $\boldsymbol{B}$ points always towards z-direction

$$\boldsymbol{B}_0 = B_0(0,0,1)$$
$$\boldsymbol{B}_1 = B_1(0,0,1)$$



Schematic representation of an atomic ensemble (blue cloud) placed in a magnetic field (green lines) in a Stern-Gerlach apparatus. From the final state the gradient of the field can be estimated.

Time evolution of the multipartite system

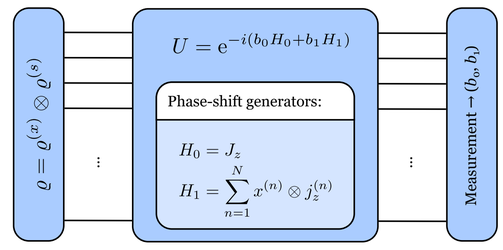
- The atoms interact only with the magnetic field ($\gamma = g\mu_\mathrm{B}$)

$$h^{(n)} = \gamma B_z^{(n)} \otimes j_z^{(n)}$$

- The collective Hamiltonian

$$H = \gamma \sum_{n=1}^{N} B_z^{(n)} \otimes j_z^{(n)}$$

- Unitary phase shift and the estimation process



Multiparametric CR bound and qFI

- The Cramér-Rao (CR) matrix inequality

$$\mathrm{Cov}(b_0, b_1) \geqslant \boldsymbol{\mathcal{F}}_\mathrm{Q}^{-1}$$

- qFI matrix elements are computed based on the generators

$$(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{i,j} = \mathcal{F}_\mathrm{Q}[\varrho, H_i, H_j]$$

$\mathcal{F}_\mathrm{Q}[\varrho, A, A] := \mathcal{F}_\mathrm{Q}[\varrho, A]$

Properties:
$\mathcal{F}_\mathrm{Q}[\varrho, A, B] = 2\sum_{i\neq j} \frac{(p_i - p_j)^2}{p_i + p_j} \langle i|A|j\rangle\langle j|B|i\rangle$
$\mathcal{F}_\mathrm{Q}[|\psi\rangle\langle\psi|, A, B] = 4(\langle AB\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi)$

Precision bound for states sensitive to the homogeneous field

Based on the CR matrix inequality the achievable precision is given by the inverted qFI matrix

$$(\Delta b_1)^{-2} \leqslant (\boldsymbol{\mathcal{F}}_\mathrm{Q})_{1,1} - \frac{(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,1}(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{1,0}}{(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,0}} \, .$$

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m}^{N} \int x_n x_m P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}, j_z^{(m)}] - \frac{(\sum_{n=1}^{N} \int x_n P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}, J_z])^2}{\mathcal{F}_\mathrm{Q}[\varrho, J_z]}$$

Precision bound for states insensitive to the homogeneous field

$$[\varrho, H_0] = 0$$

If the state is insensitive to the homogeneous field the achievable precision is given by the CR bound

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_\mathrm{Q}[\varrho, H_1].$$

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m}^{N} \int x_n x_m P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}, j_z^{(m)}]$$

The bounds are invariant under a displacement of the system

These bounds are invariant under

$$U_d = \mathrm{e}^{-idP_x}:$$

The generators in this case are

$$H_0(d) = J_z$$
$$H_1(d) = \sum_{n=1}^{N} x^{(n)} \otimes j_z^{(n)} - dJ_z \, .$$

Bound for states sensitive to the homogeneous field:

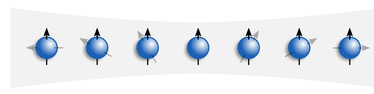
$$(\Delta b_1)^{-2} \leqslant (\boldsymbol{\mathcal{F}}_\mathrm{Q})_{1,1} - 2d(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,1} + d^2(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,0} - \frac{(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,1}^2 - 2d(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,1}(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,0} + d^2(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,0}^2}{(\boldsymbol{\mathcal{F}}_\mathrm{Q})_{0,0}}$$

Matteo G. A. Paris, Int. J. Quant. Inf. **7**, 125 (2009)

Spin chains

$$P(\boldsymbol{x}) = \prod_{n=1}^{N} \delta(x_n - na)$$

$\mu_\mathrm{ch} = a(1 + N/2)$
$\sigma_\mathrm{ch}^2 = a^2 \frac{N^2 - 1}{12}$



A chain of atoms (blue disks). All atoms are polarized perpendicular to the magnetic field. Each atom rotate with a different pace (grey arrows).

Totally polarized state $|\psi\rangle = |{+}j\rangle_y^{\otimes N}$

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m}^{N} nma^2 \, \mathcal{F}_\mathrm{Q}[|{+}j\rangle_y^{\otimes N}, j_z^{(n)}, j_z^{(m)}] - \frac{(\sum_{n=1}^{N} an \, \mathcal{F}_\mathrm{Q}[|{+}j\rangle_y^{\otimes N}, j_z^{(n)}, J_z])^2}{\mathcal{F}_\mathrm{Q}[|{+}j\rangle_y^{\otimes N}, J_z]}$$
$$= 2a^2 \frac{N^2 - 1}{12} Nj$$

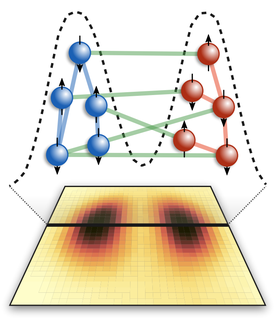
$$(\Delta b_1)^{-2} \leqslant 2\sigma_\mathrm{ch}^2 Nj$$

The bound seems to scale with the third power of N. The reason is that the length increases with the particle number. We should compare metrological usefulness of systems of the same size.

Two separated ensembles

$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n - a) \prod_{n=N/2+1}^{N} \delta(x_n + a)$$

$\mu_\mathrm{dw} = 0$
$\sigma_\mathrm{dw}^2 = a^2$



$$\int x_n P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \begin{cases} -a & \text{if } n \leqslant N/2 \\ +a & \text{if } n > N/2 \end{cases}$$

$$\int x_n x_m P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \begin{cases} a^2 & \text{if } n \text{ and } m \text{ same well} \\ -a^2 & \text{otherwise} \end{cases}$$

Atoms placed into two wells. Besides entanglement among particles belonging to the same well (red-blue lines), entanglement between the wells (green lines) can be used to overcome the shot-noise limit in gradient metrology.

G. Vitagliano *et al.*, arXiv:2104.05663 (2021)

Optimal state for gradient magnetometry

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\underbrace{+j,\ldots,+j}_{N/2}, \underbrace{-j,\ldots,-j}_{N/2}\rangle + |\underbrace{-j,\ldots,-j}_{N/2}, \underbrace{+j,\ldots,+j}_{N/2}\rangle)$$

$$(\Delta b_1)^{-2} \leqslant 4a^2N^2j^2 = 4\sigma_\mathrm{dw}^2 N^2 j^2$$

Bound for product states of pure states $|\psi\rangle \otimes |\psi\rangle$

$$(\Delta b_1)^{-2} \leqslant (\boldsymbol{\mathcal{F}}_\mathrm{Q})_{1,1}$$

$$\mathcal{F}_\mathrm{Q}[|\psi\rangle \otimes |\psi\rangle, j_z^{(n)}, j_z^{(m)}] = \begin{cases} \mathcal{F}_\mathrm{Q}[|\psi\rangle, j_z^{(n)}, j_z^{(m)}] & \text{if } n \text{ and } m \text{ same well} \\ 0 & \text{otherwise} \end{cases}$$

$$(\Delta b_1)^{-2} \leqslant 2a^2 \, \mathcal{F}_\mathrm{Q}[|\psi\rangle, J_z^{(N/2)}]$$

Essentially the homogeneous field is estimated in each of the two wells, and then the gradient is computed from the measurement results.

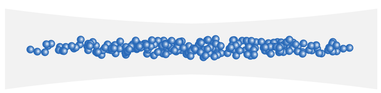
$\lvert\psi\rangle^{(N/2)}$	$\mathcal{F}_\mathrm{Q}[\lvert\psi\rangle, J_z^{(N/2)}]$	$(\Delta b_1)^{-2} \leqslant$
$\lvert\mathrm{GHZ}\rangle$	$(N/2)^2$	$a^2N^2/2$
$\lvert{+}j\rangle_y^{\otimes N/2}$	Nj	$2a^2Nj$
$\lvert\mathrm{D}\rangle_\mathrm{x}$	$N(N+4)/8$	$a^2N(N+4)/4$

Single atomic ensemble

- The probability distribution function (PDF) is permutationally invariant.

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k \in S_N} \mathcal{P}_k[P(\boldsymbol{x})]$$

$\mu = \int x_n P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$
$\sigma^2 = \int x_n^2 P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \mu^2$
$\eta = \int x_n x_m P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \mu^2$



A ensemble of atoms (blue disks). The ensemble spans along the x-direction to sense the gradient of the magnetic field.

$$\frac{-\sigma^2}{N-1} \leqslant \eta \leqslant \sigma^2$$

- We assume that the origin is at the center of the cloud of atoms, $\mu = 0$.

$$(\Delta b_1)^{-2} \leqslant \sum_{n=1}^{N} \sigma^2 \, \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}] + \sum_{n\neq m}^{N} \eta \, \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}, j_z^{(m)}]$$

$$\sum_{n,m}^{N} \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}, j_z^{(m)}] = \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, J_z]$$

$$(\Delta b_1)^{-2} \leqslant (\sigma^2 - \eta) \sum_{n=1}^{N} \mathcal{F}_\mathrm{Q}[\varrho^{(s)}, j_z^{(n)}] + \eta \underbrace{\mathcal{F}_\mathrm{Q}[\varrho^{(s)}, J_z]}_{=0 \text{ for } [\varrho, J_z]=0}$$

It indicates that if the state is insensitive to the homogeneous fields the first term might scale at most with N, so not surpassing the shot-noise limit.

On the other hand if the state is sensitive to the homogeneous fields the second term can scale with N^2 if and only if the correlation factor is positive $\eta > 0$.

Precision limit for various spin states

1. Singlet states

A pure singlet state is a simultaneous eigenstate of the collective J_z and $\boldsymbol{J}^2$ operators, with an eigenvalue zero for both operators.

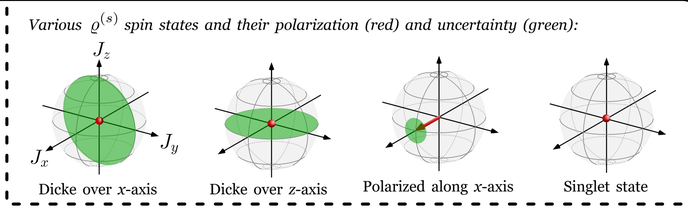
For a mixed singlet:

$$\langle 0,0,D|j_z^{(n)}|0,0,D'\rangle = 0 \, .$$

$$\mathcal{F}_\mathrm{Q}[\varrho_\mathrm{singlet}, j_z^{(n)}] = 4\langle j_z^{(n)2}\rangle$$

$$(\Delta b_1)^{-2} \leqslant (\sigma^2 - \eta) N \frac{4j(j+1)}{3}$$

Various $\varrho^{(s)}$ spin states and their polarization (red) and uncertainty (green):



2. Totally polarized state along y-axis

$$\mathcal{F}_\mathrm{Q}[|{+}j\rangle_y^{\otimes N}, j_z^{(n)}] = 2j \qquad \mathcal{F}_\mathrm{Q}[|{+}j\rangle_y^{\otimes N}, J_z] = 2Nj$$

$$(\Delta b_1)^{-2} \leqslant 2\sigma^2 Nj$$

3. Best separable state

$$\mathcal{F}_\mathrm{Q}[\varrho, j_z^{(n)}, j_z^{(m)}] = \begin{cases} 0 & \text{if } n \neq m, \\ 4(\Delta j_z^{(n)})^2 & \text{if } n = m. \end{cases}$$

$$(\Delta b_1)^{-2} \leqslant 4\sigma^2 Nj^2$$

4. Unpolarized Dicke states

$|\mathrm{D}\rangle_l$ is a simultaneous eigenstate of the collective J_l and $\boldsymbol{J}^2$ operators, with an eigenvalue zero and $Nj(Nj+1)$, respectively.

For spin 1/2 systems we have that

$$\langle j_x^{(n)2}\rangle = \langle j_y^{(n)2}\rangle = \langle j_z^{(n)2}\rangle = \frac{1}{4}.$$

$$\mathcal{F}_\mathrm{Q}[|\mathrm{D}\rangle_l, j_z^{(n)}] = 1 \qquad \mathcal{F}_\mathrm{Q}[|\mathrm{D}\rangle_x, J_z] = \frac{N}{2}(N+2)$$

$$(\Delta b_1)^{-2} \leqslant (\sigma^2 - \eta)N + \underbrace{\eta \frac{N(N+2)}{2}}_{=0 \text{ if for } [\varrho, J_z]=0}$$

5. GHZ states

Such states are very sensitive to the homogeneous field.

$$\langle j_z^{(n)2}\rangle = \frac{1}{4} \qquad \langle J_z\rangle = \frac{N^2}{4}$$

$$(\Delta b_1)^{-2} \leqslant (\sigma^2 - \eta)N + \eta N^2$$

Conclusions

- We obtained precision limits of estimating the gradient of a magnetic field with atomic ensembles initialized in different states.
- For the two-ensemble case, the precision of the estimation of the gradient can reach the Heisenberg limit.
- For a single ensemble with localized particles, the shot-noise limit can be surpassed if there is a strong correlation between the particle positions.
- Single-ensemble methods can have a huge practical advantage compared to methods based on two or more atomic ensembles since using a single ensemble makes the experiment simpler and can also result in a better spatial resolution.

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