Gradient magnetometry with various types of spin ensembles

Single atomic ensembles, chain of spins & two different ensembles

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Información Cuántica en España (2023)
- May 30, 2023 -

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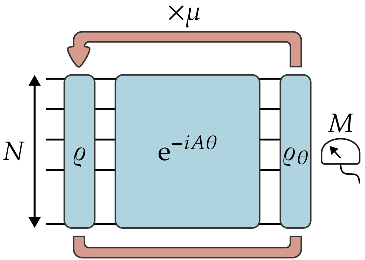
Outline



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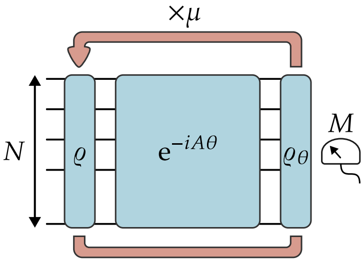
$$\frac{1}{(\Delta\theta)^2} \leqslant \mu\mathcal{F}_{\mathrm{Q}}[\varrho, A].$$



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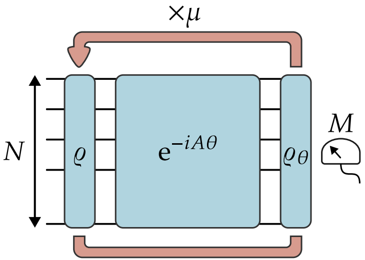
Quantum Metrology

Quantum Fisher Information



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$$\mathcal{F}_Q[\varrho, A] = 2\sum_{\lambda\neq\mu} \frac{(p_\lambda - p_\mu)^2}{p_\lambda + p_\mu} |\langle\lambda|A|\mu\rangle|^2$$

written on the eigenbasis of the state, $\varrho = \sum p_\lambda |\lambda\rangle\langle\lambda|$.

[M.G.A. Paris (2009), IJQI **7**, 125]

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Properties of the qFI for a single parameter estimation problem

1. It is independent of the measurement. An optimal measurement exists though, which saturates the qCR bound.

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Quantum Metrology

QFI and entanglement

Entanglement: where $U = \exp(-i\theta J_z)$ and $J_z = \sum_n^N j_z^{(n)}$.

1. Separable states can achieve at most the so-called Shot-noise limit (SNL),

$$\mathcal{F}_Q[\varrho_{\text{sep}}, J_z] \leqslant N.$$

2. An ultimate limit is obtained maximizing the qFI over all pure states

$$\max_{|\Psi\rangle} \mathcal{F}_Q[|\Psi\rangle, J_z] = N^2,$$

which is called the Heisenberg limit.
Hence, entanglement is needed to overcome the SNL.

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E.g. entanglement criteria based on qFI

- Due to its tight relation with the variance, qFI has been used to improve some entanglement conditions.

[G Tóth (2022), PRR **4** 013075]

- Ion chains can be used to estimate the magnetic field as a function of position, $B(x)$.

[Matteo Fadel *et al.*, arXiv.org:2201.11081]

Multiparameter Quantum Metrology

Motivation

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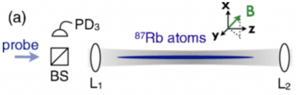
- States insensitive to the homogeneous fields have been prepared in elongated traps.



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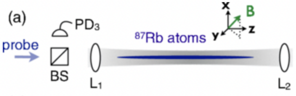

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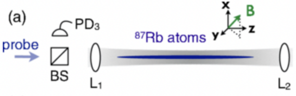

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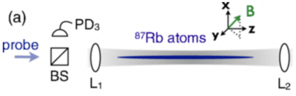

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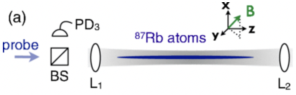

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We want to estimate B_1.

Consider the following evolution for the state

$$\varrho_{\boldsymbol{\theta}} = \mathrm{e}^{-i\sum_k A_k\theta_k}\,\varrho\,\mathrm{e}^{+i\sum_k A_k\theta_k}\,.$$

- In this case the CR bound is a matrix inequality for the covariance matrix

$$\mathrm{Cov}[\theta_i,\theta_j] \geqslant \frac{1}{\mu}(\mathcal{F}_{\mathrm{Q}}{}^{-1})_{i,j},$$

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- When $[A_i, A_j] = 0$, the bounds can be saturated.

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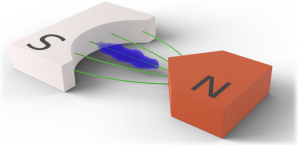
Outline

Gradient magnetometry and basic setup



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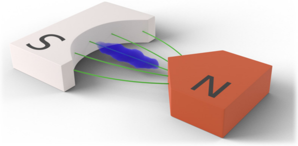

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- In this work we assume that the position state is an statistical mixture of point-like particles

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- The two unknown parameters are B_0 and B_1 are encoded in b_0 and b_1 acting onto the state with the following unitary operator

$$U = \mathrm{e}^{-i(b_0 H_0 + b_1 H_1)},$$

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In the following we are interested on the precision bound for b_1, the gradient parameter.

Gradient magnetometry and basic setup

Precision bounds for states **insensitive** to the homogeneous B_0

For states that commute with the homogeneous field, $[\varrho, J_z] = 0$, the precision bound is

$$\frac{1}{(\Delta b_1)^2} \leqslant \mathcal{F}_{\mathrm{Q}}[\varrho, H_1],$$

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Outline

1. Multiparameter Quantum Metrology
   - Cramér-Rao precision bound and quantum Fisher information
   - Multiparameter qFI matrix and simultaneous estimation
2. System setup and precision bounds of the gradient parameter estimation for various states
   - Gradient magnetometry and basic setup of the system
   - Precision bounds for various systems and different spin states
3. Conclusions

Chain of qubits

$$P(\boldsymbol{x}) = \prod_{n} \delta(x_n - na).$$

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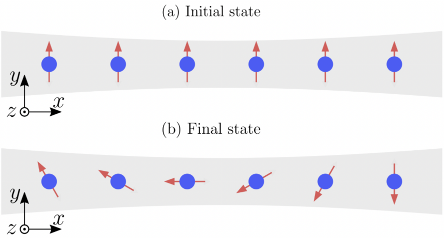
Totally polarized $|0\rangle_y^{\otimes N}$ state under a magnetic field pointing towards the z-direction



$$\frac{1}{(\Delta b_1)^2} \leqslant \sum_{n,m} nma^2 \mathcal{F}_Q[|0\rangle_y^{\otimes N}, j_z{}^{(n)}, j_z{}^{(m)}] - \frac{\left(\sum_n na\mathcal{F}_Q[|0\rangle_y^{\otimes N}, j_z{}^{(n)}, J_z]\right)^2}{\mathcal{F}_Q[|0\rangle_y^{\otimes N}, J_z]}$$

Chain of qubits

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- Mean particle position:

$$\mu = a\frac{N+1}{2}$$

- Variance of the particle positions:

$$\sigma^2 = a^2\frac{N^2-1}{12}$$

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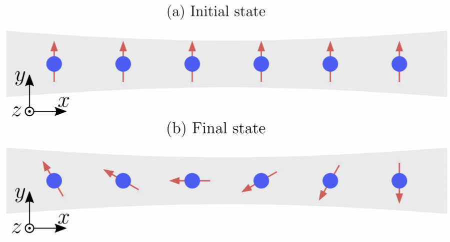

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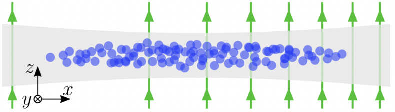
$$= a^2\frac{N^2-1}{12}N = \sigma^2 N.$$

Single ensemble of point-like spin-$\frac{1}{2}$ atoms

Permutationally invariant PDF

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k \in S_N} \mathcal{P}_k[P(\boldsymbol{x})]$$



[N Behbood *et al.* (2014), PRL **113** 093601]

- $\mu = \int x_n P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$.
- $\sigma^2 = \int x_n^2 P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, if the origin is at 0.
- $\eta = \int x_n x_m P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ for $n \neq m$.
  $\eta \in [-\sigma^2/(N-1), \sigma^2]$.

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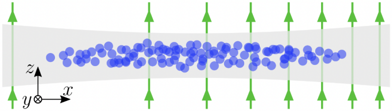

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Precision CR bound

$$\frac{1}{(\Delta b_1)^2} \leqslant (\sigma^2 - \eta) \sum_n \mathcal{F}_Q[\varrho^{(s)}, {j_z}^{(n)}] + \eta\, \mathcal{F}_Q[\varrho^{(s)}, J_z]$$

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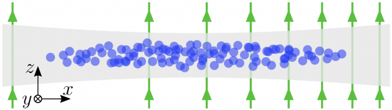
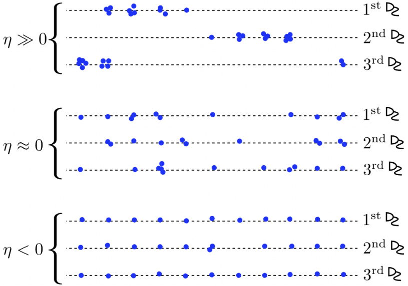
Permutationally invariant PDF

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k \in S_N} \mathcal{P}_k[P(\boldsymbol{x})]$$



[N Behbood *et al.* (2014), PRL **113** 093601]

- $\mu = \int x_n P(\boldsymbol{x})\, d\boldsymbol{x}$.
- $\sigma^2 = \int x_n^2 P(\boldsymbol{x})\, d\boldsymbol{x}$, if the origin is at 0.
- $\eta = \int x_n x_m P(\boldsymbol{x})\, d\boldsymbol{x}$ for $n \neq m$.
  $\eta \in [-\sigma^2/(N-1), \sigma^2]$.



Precision CR bound

$$\frac{1}{(\Delta b_1)^2} \leqslant (\sigma^2 - \eta) \sum_n \mathcal{F}_Q[\varrho^{(s)}, {j_z}^{(n)}] + \eta\, \mathcal{F}_Q[\varrho^{(s)}, J_z]$$

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Singlet states

$$\varrho^{(\mathrm{s})} = \sum_{\lambda} p_{\lambda} |0, 0, i\rangle\langle 0, 0, i|$$

Its precision bound is

$$\frac{1}{(\Delta b_1)^2} \leqslant (\sigma^2 - \eta)N.$$

Single ensemble of point-like spin-$\frac{1}{2}$ atoms

Precision bounds for various spin states

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Best separable state

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle_{\mathrm{sep}}, j_z{}^{(n)}, j_z{}^{(m)}] = \begin{cases} 4(\Delta j_z{}^{(n)})^2 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Then, the precision is

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$|\mathrm{GHZ}\rangle$ states

$$\mathcal{F}_{\mathrm{Q}}[|\mathrm{GHZ}\rangle, {j_z}^{(n)}] = 1$$

and

$$\mathcal{F}_{\mathrm{Q}}[|\mathrm{GHZ}\rangle_x, J_z] = N^2.$$

Hence,

$$\frac{1}{(\Delta b_1)^2} \leqslant (\sigma^2 - \eta)N + \eta N^2.$$

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*More in PRA **97**, 053603 (2018)*

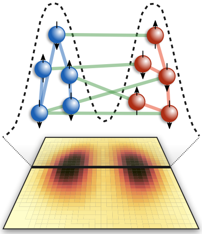
Double well of atoms

$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n + a) \prod_{n=N/2+1}^{N} \delta(x_n - a)$$

The contribution of the position of the particles:

$$\int x_n P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \begin{cases} -a \\ +a \end{cases} \quad \text{and} \quad \int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \begin{cases} +a^2 \\ -a^2 \end{cases}$$

- In this case the mean position is $\mu = 0$ and the variance is $\sigma^2 = a^2$.



[K Langle *et al.* (2018), Science **360** 6387]

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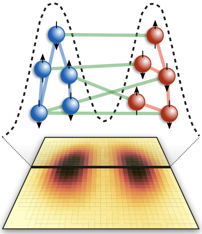

[K Langle *et al.* (2018), Science **360** 6387]

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For spin-$\frac{1}{2}$ system, the state that maximizes the bound is

$$|\psi\rangle = \frac{|\overbrace{0,\dots,0}^{N/2},\overbrace{1,\dots,1}^{N/2}\rangle + |1,\dots,1,0,\dots,0\rangle}{\sqrt{2}}, \qquad \text{and} \qquad \frac{1}{(\Delta b_1)^2} \leqslant \sigma^2 N^2.$$

System setup and Δb_1 bounds

Double well of atoms

Product of two equal spin states

For states of the type $|\psi\rangle^{(\mathrm{L})} \otimes |\psi\rangle^{(\mathrm{R})}$, we have that

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})} \otimes |\psi\rangle^{(\mathrm{R})}, {j_z}^{(n)}, {j_z}^{(m)}] = \begin{cases} \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, {j_z}^{(n)}, {j_z}^{(m)}] & \text{if } n \text{ and } m \text{ same well} \\ 0 & \text{otherwise} \end{cases}$$

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Hence, the precision bounds can be simply computed for $N/2$ particles at one of the wells,

$$\frac{1}{(\Delta b_1)^2} \leqslant 2\sigma^2 \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, J_z{}^{(N/2)}] \leqslant \sigma^2 N^2 /2.$$

Double well of atoms

What if we want to estimate both parameters?

If we assume $a = 1$, we have that $H_0 = {J_z}^{(\mathrm{L})} + {J_z}^{(\mathrm{R})}$ and $H_1 = {J_z}^{(\mathrm{L})} - {J_z}^{(\mathrm{R})}$.

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2\mathcal{F}_{\mathrm{Q}}[\rho, {J_z}^{(\mathrm{L})}] + 2\mathcal{F}_{\mathrm{Q}}[\rho, {J_z}^{(\mathrm{R})}]$$

Separable states

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2N_{\mathrm{L}} + 2N_{\mathrm{R}} = 2N.$$

Heisenberg limit for evenly split systems

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2N_{\mathrm{L}}^2 + 2N_{\mathrm{R}}^2 = N^2.$$

Examples

$$|\mathrm{GHZ}\rangle \quad \rightarrow \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_0] = N^2 \quad \text{and} \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_1] = 0.$$

$$|\psi\rangle = \frac{|\overbrace{0,\ldots}^{N/2},\overbrace{1,\ldots}^{N/2}\rangle + |1,\ldots,0,\ldots\rangle}{\sqrt{2}} \quad \rightarrow \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_1] = N^2 \quad \text{and} \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_0] = 0.$$

Conclusions

- In principle, the **effect of an unknown global rotation** has to be considered.

- For a **single ensemble** with localized particles, a method with a huge practical advantage, the shot-noise limit can be surpassed if and only if there is a **strong statistical correlation between the particle positions**.

- There is a **trade-off** between homogeneous and gradient magnetometry if one wants to estimate both parameters at the same time.

Summary

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Thank you for your attention!