Multicopy metrology with many-particle quantum states

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Entanglement

An N-partite quantum state is entangled if it cannot be written as

$$\varrho = \sum_i p_i \varrho_i^{(A_1)} \otimes \varrho_i^{(A_2)} \cdots \otimes \varrho_i^{(A_N)}.$$

Required for quantum advantage

- Quantum teleportation
- Superdense coding
- Quantum secure communication
- Quantum metrology

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Basic task in quantum metrology

$$\varrho \Rightarrow \overset{\text{Linear interferometer}}{\boxed{U_\theta = \exp(-i\mathcal{H}\theta)}} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \overset{\text{Quantum measurement}}{\boxed{\text{Estimation of } \theta}}$$

- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N$$

where h_n's are single-subsystem operators.

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  where h_n's are single-subsystem operators.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}_Q[\varrho, \mathcal{H}]},$$

  where the quantum Fisher information is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2,$$

  with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition.

Metrological gain

- For a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

where the separable limit for *local* Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

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- $g_{\mathcal{H}}(\varrho)$ can be maximized over *local* Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho).$$

- If $g(\varrho) > 1$ then the state is useful metrologically.
[G. Tóth et al., PRL 125, 020402 (2020)]

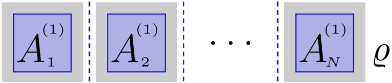
Motivation

- Entanglement is required for usefulness
- Some highly entangled (pure) states are not useful
  [P. Hyllus et al., PRA 82, 012337 (2010)]
- But some weakly entangled states can be useful
  [G. Tóth and T. Vértesi, PRL 120, 020506 (2018)]
- What kind of states can be made useful with extended techniques?

Outline

The considered setting

Can considering more copies of a state ϱ help?



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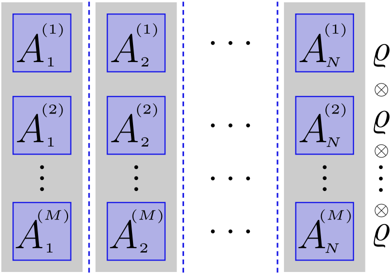

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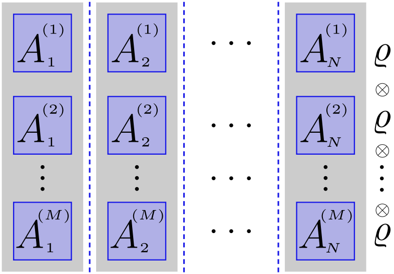

Can we have $g(\varrho^{\otimes M}) > 1 \geq g(\varrho)$?
[G. Tóth et al., PRL 125, 020402 (2020)]

A special subspace

Observation

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

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- *Proof.*—Consider a state from this subspace

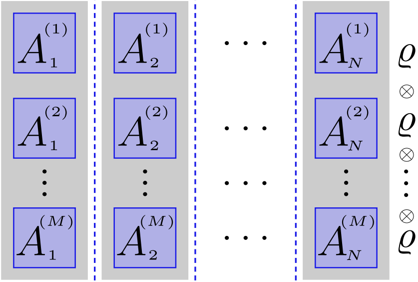
$$\varrho = \sum_{k,l=0}^{d-1} c_{kl}(|k\rangle\langle l|)^{\otimes N}$$

- Using the relation

$$\mathcal{F}_Q[\varrho, \mathcal{H}] \geq 4I_\varrho(\mathcal{H}) = 4\left[\mathrm{Tr}(\varrho\mathcal{H}^2) - \mathrm{Tr}(\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}\mathcal{H})\right].$$

- Computing $I_{\varrho^{\otimes M}}(\mathcal{H})$ with $\mathcal{H} = \sum_{n=1}^{N}(D^{\otimes M})_{A_n}$, where $D = \mathrm{diag}(+1, -1, +1, -1, ...)$.

A special subspace



$$
\begin{array}{lcccccccc}
h_1 = & D^{\otimes M} & \otimes & \mathbb{1} & \otimes & \cdots & \otimes & \mathbb{1} \\
h_2 = & \mathbb{1} & \otimes & D^{\otimes M} & \otimes & \cdots & \otimes & \mathbb{1} \\
\vdots & \vdots & & \vdots & & & & \vdots \\
h_n = & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \cdots & \otimes & D^{\otimes M}
\end{array}
$$

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Examples

- The state

$$p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2},$$

with $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

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- For M copies of the state

$$\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2} + c_{01}(|0\rangle\langle 1|)^{\otimes N} + c_{01}^{*}(|1\rangle\langle 0|)^{\otimes N},$$

  we have

$$4I(c_{01}, N) = 4N^2[1 - (1 - 4|c_{01}|^2)^{M/2}].$$

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- All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k\,|k\rangle^{\otimes N}.$$

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Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

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- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}| + (1-p)\mathbb{1}/2^2,$$

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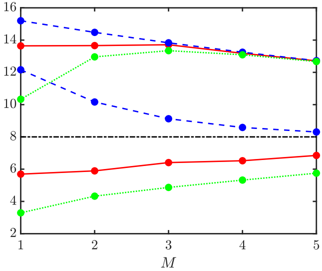
- $\varrho^{(0.9)}$ (top 3 curves) and $\varrho^{(0.52)}$ (bottom 3 curves). $h_n = \sigma_z^{\otimes M}$.

$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho, \mathcal{H}] \geq 4I_\varrho(\mathcal{H})$



Outline

"GHZ"-like states

Observation

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

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- The state for $N \geq 3$

$$|\psi\rangle = \sigma_0 \left|0\right\rangle^{\otimes N} + \sigma_1 \left|1\right\rangle^{\otimes N}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].

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- But

$$\left|\psi'\right\rangle = \sigma_0 \left|0\right\rangle^{\otimes N} + \sigma_1 \left|1\right\rangle^{\otimes N} + 0\left|2\right\rangle^{\otimes N}$$

  is always useful.
- The non-useful $\left|\psi\right\rangle$, embedded into $d = 3$ ($\left|\psi'\right\rangle$) becomes useful.

Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

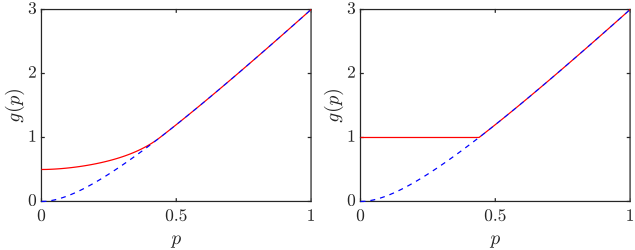

Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

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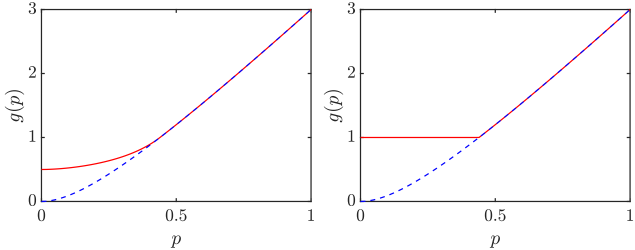

Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

- $\varrho_3^{(p)}$ is genuine multipartite entangled for $p > 0.428571$ [SM Hashemi Rafsanjani et al., PRA 86, 062303 (2012)].
- $\varrho_3^{(p)}$ is useful metrologically for $p > 0.439576$.

Outline

Scaling for a special pure state

- For separable states $g_{\mathcal{H}} \sim 1$ ($\mathcal{F}_Q \sim N$) at best (shot-noise scaling).
- For entangled states $g_{\mathcal{H}} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best (Heisenberg scaling).

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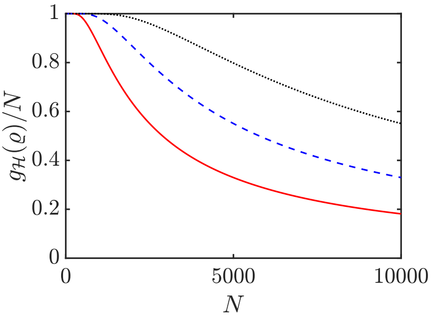

Figure: Dependence of the metrological gain on the particle number N for (solid) $M = 2000$, (dashed) 4000 and (dotted) 6000 copies. $h_n = \sigma_z^{\otimes M}$

Conclusions

- Investigated metrological performance of different quantum states when we have more copies of them.
- Identified a subspace in which all the states become useful if sufficiently many copies are taken.
- Also improved metrological performance by embedding.

See arXiv:2203.05538 (2022)!

Thank you for the attention!

QUANTERA
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