Activation of metrologically useful genuine multipartite entanglement

arXiv:2203.05538 (2022)

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   - Quantum metrology
2. Improving metrological performance
   - Taking many copies
   - Embedding into higher dimension

Basic task in quantum metrology

Linear interferometer — Quantum measurement

$$\varrho \Rightarrow \boxed{U_\theta = \exp(-i\mathcal{H}\theta)} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \boxed{\text{Estimation of } \theta}$$

- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N$$

where h_n's are single-subsystem operators of the N-partite system.

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  where h_n's are single-subsystem operators of the N-partite system.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}_Q[\varrho, \mathcal{H}]},$$

  where the quantum Fisher information is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2,$$

  with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition.

Metrological gain

- For a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

where the separable limit for *local* Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

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- $g_{\mathcal{H}}(\varrho)$ can be maximized over *local* Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho).$$

- If $g(\varrho) > 1$ then the state is useful metrologically.
[G. Tóth et al., PRL 125, 020402 (2020)]

Relation to multipartite entanglement

- Fully-separable states $\rightarrow g \leq 1$ (shot-noise scaling).
- Entanglement is required for usefulness.
- Even weakly entangled states can be useful

  [G. Tóth and T. Vértesi, PRL 120, 020506 (2018)]
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- $g > k \rightarrow$ *metrologically useful* $(k+1)$-partite entanglement.
- $g > N-1 \rightarrow$ *metrologically useful* N-partite/genuine multipartite entanglement (GME).
- $g = N$ ($\mathcal{F}_Q = 4N^2$) is the maximal usefulness (Heisenberg scaling).

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- $g = N$ ($\mathcal{F}_Q = 4N^2$) is the maximal usefulness (Heisenberg scaling).
- There are non-useful GME states [P. Hyllus et al., PRA 82, 012337 (2010)]
- What kind of entangled states can be made useful with extended techniques?

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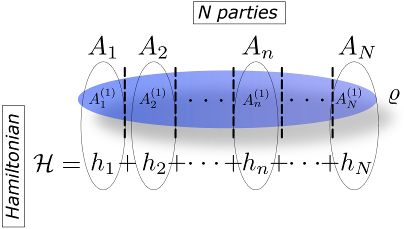
The considered setting

Can considering more copies of the N-partite state ϱ help?



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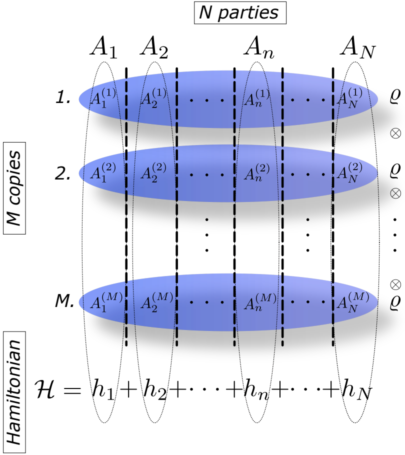

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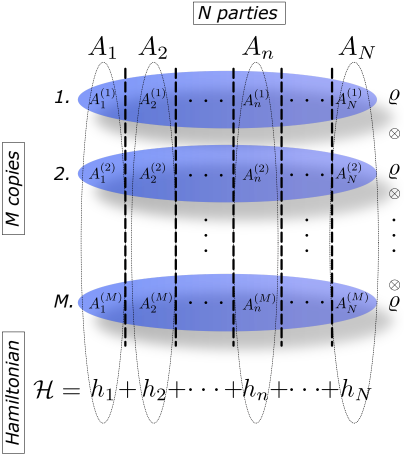

Can we have $g(\varrho^{\otimes M}) > 1 \geq g(\varrho)$? [G. Tóth et al., PRL 125, 020402 (2020)]

Metrologically useful GME activation

Result

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

The maximum is attained exponentially fast with the number of copies.

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$$h_n = D^{\otimes M}, \text{ for } 1 \leq n \leq N$$

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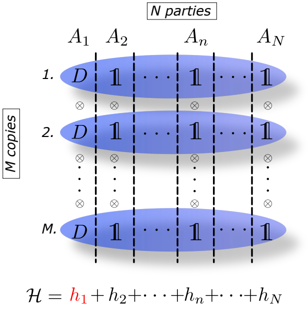

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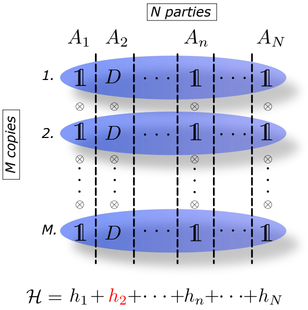

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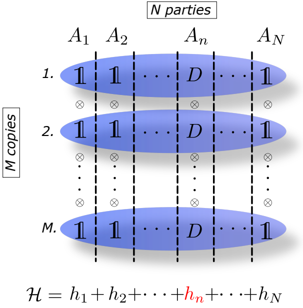

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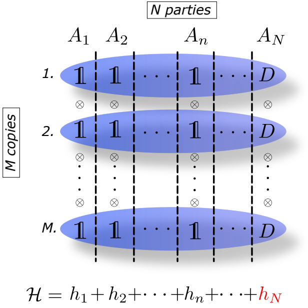

Examples

- The state with $|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$

$$\varrho_N(p) = p\,|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2},$$

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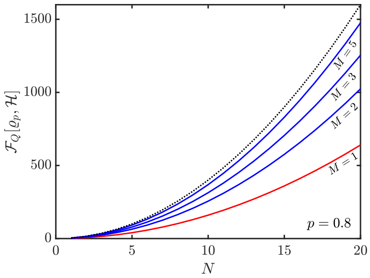

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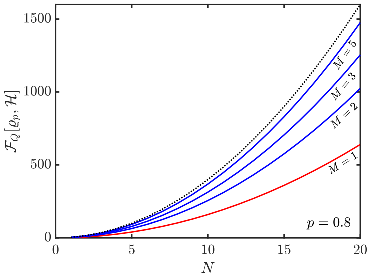

- All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k\,|k\rangle^{\otimes N}.$$

Optimal measurements

- In the limit of many copies ($M \gg 1$)

$$\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 4N^2 \implies (\Delta\theta)^2 \geq 1/\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 1/4N^2$$

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$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{|\partial_\theta \langle\mathcal{M}\rangle|^2} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2}.$$

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- For $M = 2$ copies of $\varrho_3(p)$

$$\begin{aligned}\mathcal{M} =& \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \\ &+ \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y\end{aligned}$$

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“GHZ”-like states

Result

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

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- The state for $N \geq 3$ with $d = 2$

$$\left|\psi\right\rangle = \sigma_0 \left|0\right\rangle^{\otimes N} + \sigma_1 \left|1\right\rangle^{\otimes N}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].

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- But with $d = 3$

$$|\psi'\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0 |2\rangle^{\otimes N}$$

  is always useful.

- The non-useful $|\psi\rangle$, embedded into $d = 3$ ($|\psi'\rangle$) becomes useful.

Conclusions

- Investigated the metrological performance of quantum states in the multicopy scenario.
- Identified a subspace in which metrologically useful GME activation is possible.
- Also improved metrological performance by embedding.

See arXiv:2203.05538 (2022)!

Thank you for the attention!

QUANTERA

Alexander von Humboldt Stiftung/Foundation

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EUSKO JAURLARITZA GOBIERNO VASCO

NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE

The general measurements for Observation 1

$$\varrho(p,q,r) = p\,|\mathrm{GHZ}_q\rangle\langle\mathrm{GHZ}_q| + (1-p)[r(|0\rangle\langle 0|)^{\otimes N} + (1-r)(|1\rangle\langle 1|)^{\otimes N}],$$

with

$$|\mathrm{GHZ}_q\rangle = \sqrt{q}\,|000..00\rangle + \sqrt{1-q}\,|111..11\rangle\,,$$

The following operator, being the sum of M correlation terms

$$\mathcal{M} = \sum_{m=1}^{M} Z^{\otimes(m-1)} \otimes Y \otimes Z^{\otimes(M-m)},$$

where we define the operators acting on a single copy

$$Y = \begin{cases} \sigma_y^{\otimes N} & \text{for odd } N, \\ \sigma_x \otimes \sigma_y^{\otimes(N-1)} & \text{for even } N, \end{cases}$$

$$Z = \sigma_z \otimes \mathbb{1}^{\otimes(N-1)}.$$

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1/[4q(1-q)] + (M-1)p^2}{4MN^2p^2}.$$

Observation

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

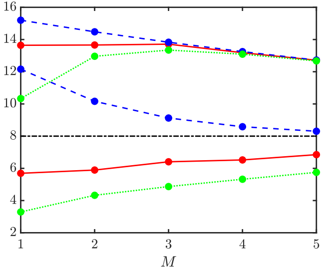
- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\rm me}\rangle\langle\Psi_{\rm me}| + (1-p)\mathbb{1}/2^2,$$

where $|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

- $\varrho^{(0.9)}$ (top 3 curves) and $\varrho^{(0.52)}$ (bottom 3 curves). $h_n = \sigma_z^{\otimes M}$.

$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H})$



Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

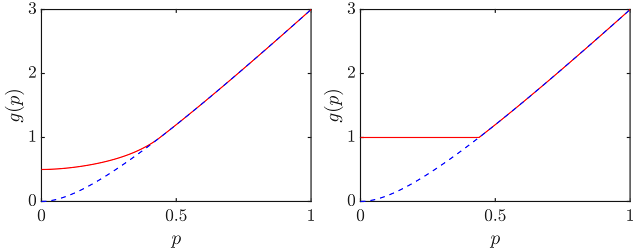


Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

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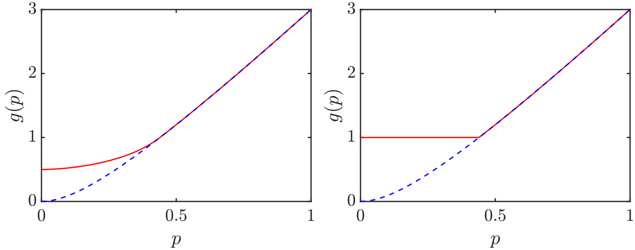

Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

- $\varrho_3^{(p)}$ is genuine multipartite entangled for $p > 0.428571$ [SM Hashemi Rafsanjani et al., PRA 86, 062303 (2012)].
- $\varrho_3^{(p)}$ is useful metrologically for $p > 0.439576$.