Variational-state quantum metrology

and

Continuous phase-space representations for qubit and qudit systems

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Introduction

- metrology: measurement precision of a quantity
- QM: information encoded in quantum states
- measurement: fundamental limitations on precision
- quantum information sensitive to noise

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aim is to find the best quantum states for metrology using quantum computers

- variational algorithms: efficiently explore Hilbert space
- expected to be first applications of quantum computers
- quantum chemistry (VQE) or machine learning

Basic setup in quantum metrology

qubit state $|\psi\rangle$ evolves under the Hamiltonian

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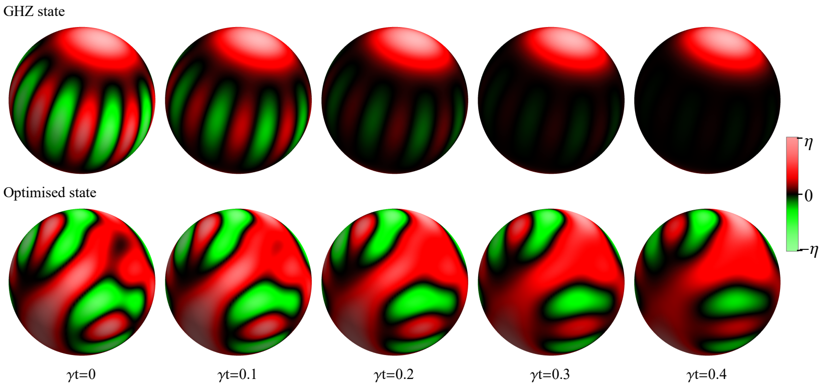
- measurements of an observable $O = \sum_{n=1}^{d} \lambda_n |n\rangle\langle n|$
- results in probabilities $p(n|\omega)$ that depend on ω
- precision is determined by classical Fisher information

$$F_c(O) = \sum_n p(n|\omega)\left(\frac{\partial \ln\, p(n|\omega)}{\partial\omega}\right)^2$$

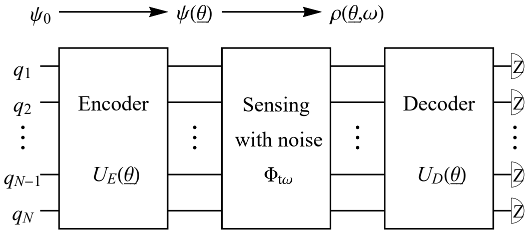
Effect of noise

- GHZ states are optimal $(|000\cdots0\rangle + |111\cdots1\rangle)/\sqrt{2}$
- but very sensitive to noise: super decoherence
- optimal states: robust to noise and sensitive to field



Variational-state quantum metrology

- parametrised probe state $|\psi(\underline{\theta})\rangle$ via encoder
- estimate and optimise metrological usefulness of $|\psi(\underline{\theta})\rangle$:
    - interaction with external field ω under noise
    - precision of ω: Fisher information via measurements



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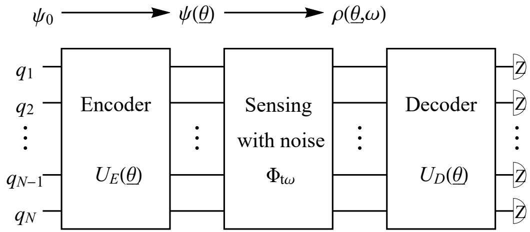
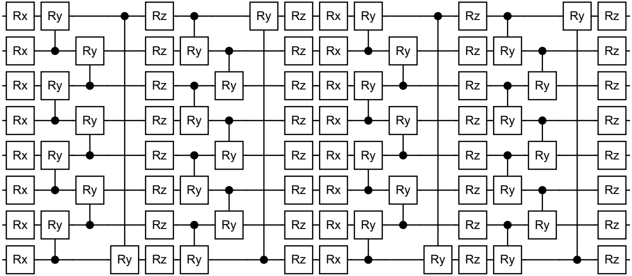


- optimal measurement basis O via decoder
- can be implemented on near-term quantum hardware

Encoder circuit generates probe states

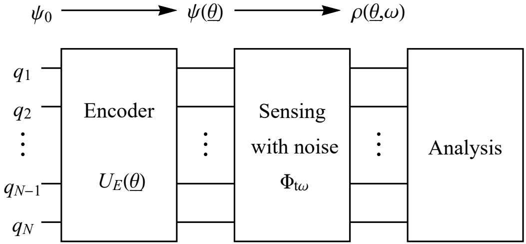
- encoder as a quantum circuit with quantum gates
- every gate is parametrised: optimise parameters



- efficient search: linear number of parameters (here 80)
- ansatz states: (general) not permutation symmetric
- good approximation of metrologically optimal states

Numerical simulations

- exact simulation of quantum circuits in QuEST
- general noise via Kraus operators $\Phi_{\omega t}(\rho) = e^{-i\omega t \mathcal{J}_z + \gamma t \mathcal{L}}\rho$



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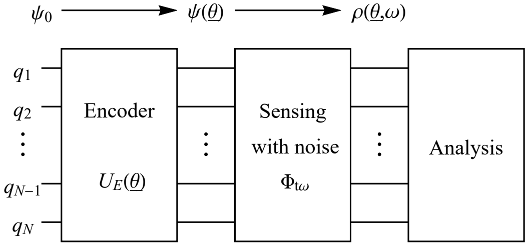
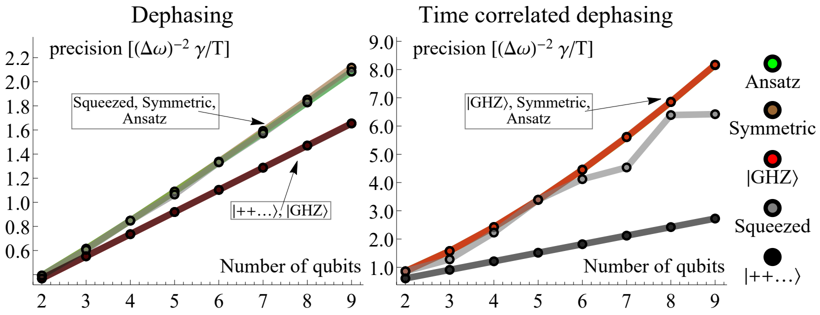


- calculate *dimensionless* precision $\gamma/T(\Delta\omega)^{-2}_{\max}$
- via quantum Fisher information of $\rho(\omega t, \underline{\theta})$
- optimise parameters $\underline{\theta}$: up to 9 qubits (2 weeks)

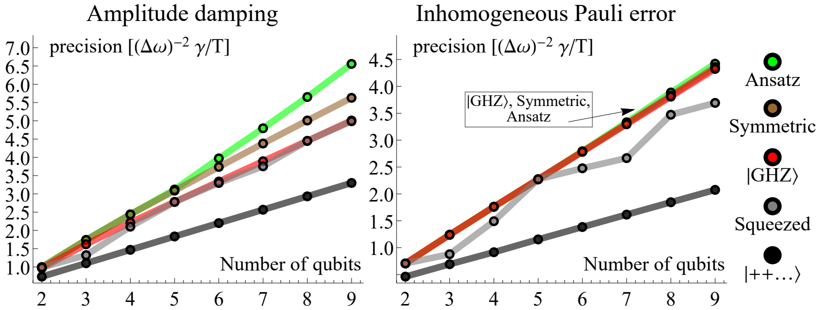
Error models

- simulated various different error models
- comprehensively explored up to 9 qubits
- dephasing and non-Markovian noise: simple solution
- known solutions – symmetric states are optimal



Results

- comparing ansatz states to previously known ones
- previous assumption: symmetric states are optimal
- brown: direct search in symmetric subspace
- significant improvement of precision with ansatz states

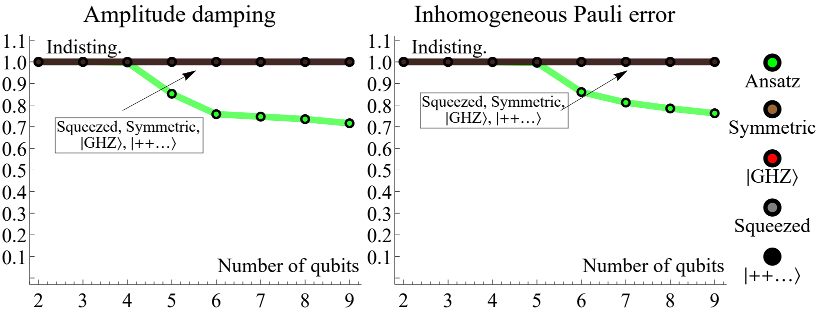


Broken permutation symmetry

- calculating a measure of indistinguishability $P_{\text{avg}}(|\psi\rangle)$

$$P_{\text{avg}}(|\psi\rangle) := \frac{1}{N_p} \sum_{k=1}^{N_p} \text{Fid}[|\psi\rangle, P_k|\psi\rangle]$$

- optimal states have broken permutation symmetry



Analytical model for amplitude damping

- optimal states: $c_1|11\cdots 1\rangle + c_2|D\rangle + c_3|00\cdots 0\rangle$
- $|D\rangle$ can passively correct (firs-order) decay events

$$|D\rangle = \sqrt{\frac{2}{N}}(|1100\cdots 000\rangle + |0011\cdots 000\rangle \cdots + |0000\cdots 011\rangle)$$

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$$|D\rangle = \sqrt{\tfrac{2}{N}}(|1100\cdots000\rangle + |0011\cdots000\rangle \cdots + |0000\cdots011\rangle)$$

- contains double excitations – flips of qubit pairs
- can passively correct first-order decay events
- optimal measurement basis resolves individual flips
- superior to its symmetric counterpart $|J, J-2\rangle$

Summary

- variational algorithms are potentially powerful
- here: application to quantum metrology
- can be implemented on near-term quantum hardware
- numerical simulations reveal interesting features
- symmetry breaking of optimal states
- analytical model of symmetry breaking

Part II.: phase-space representations

aim is to represent and analyse quantum states in phase space

- plethora of techniques: variants of the Wigner function
  - quantum optics: s-parametrized family
  - time-frequency analysis: Born-Jordan distribution
- applications include: tomography, efficient comp.

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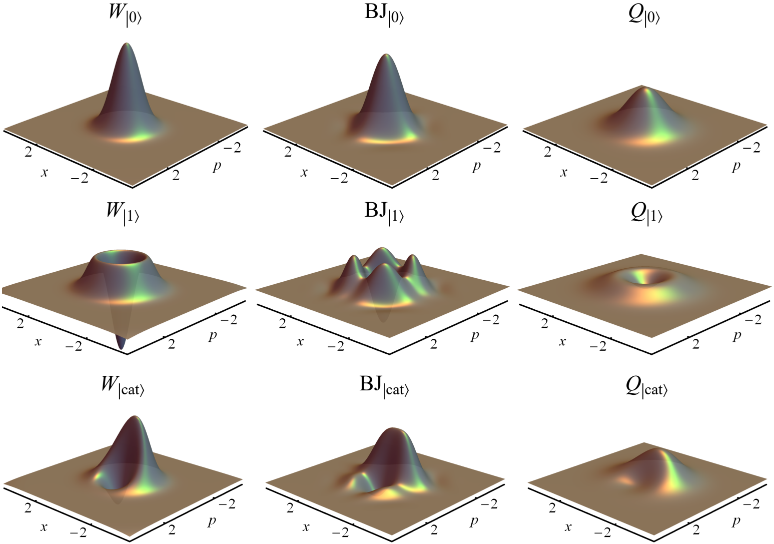
- flat phase space can be generalised to manifolds
- spherical phase space of qubits and qudits (spins)

Example phase-space plots



B. Koczor, F. vom Ende, M. A. de Gosson, S. J. Glaser, R. Zeier:
Phase Spaces, Parity Operators, and the Born-Jordan Distribution.
(2018) arXiv:1811.05872

Infinite-dimensional systems: Wigner's original approach

$$W_\psi(x,p) = (2\pi\hbar)^{-1} \int e^{-\frac{i}{\hbar}py} \psi^*(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y)\, \mathrm{d}y$$

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Theorem:

$$F_\rho(\Omega, \theta) = (\pi\hbar)^{-1} \, \mathrm{tr}\, [\, \rho\, \mathcal{D}(\Omega) \Pi_\theta \mathcal{D}^\dagger(\Omega)]$$

- Cohen class as convolutions: $F_\rho(\Omega, \theta) \equiv \theta(\Omega) * W_\rho(\Omega)$
- e.g., Wigner function if θ is the delta distribution
- more special cases: Husimi Q, Glauber P, Born-Jordan

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- more special cases: Husimi Q, Glauber P, Born-Jordan
- parity operators Π_θ, e.g., $\Pi|\Omega\rangle = |-\Omega\rangle$
- flat phase-space coordinates $\Omega \simeq (x, p) \simeq \alpha$
- translation symmetry: displacements $\mathcal{D}(\Omega)|0\rangle = |\Omega\rangle$

The Born-Jordan distribution

- very interesting special case: Born-Jordan distribution
- fundamental importance in quantisation

Theorem: Born-Jordan parity operator via squeezing

$$\Pi_{\mathrm{BJ}} = \left[\, \tfrac{1}{4} \int_{-\infty}^{\infty} \mathrm{sech}(\tfrac{\xi}{2}) S(\xi)\, \mathrm{d}\xi \,\right] \Pi,$$

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- one-parameter squeezing unitary with $\xi \in \mathbb{R}$

$$S(\xi)\psi(x) = e^{\xi/2}\, \psi(e^{\xi} x)$$

- composition of squeezing and coordinate reflection Π

Spectral decomposition and matrix representation

purely continuous spectrum: not square-integrable $|\psi_{\pm}^{E}\rangle$

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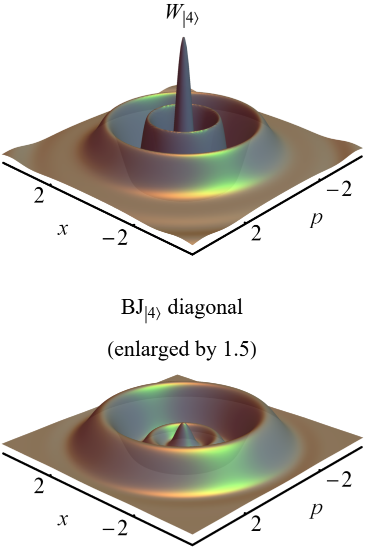
matrix elements in the number-state basis – finite sum

Theorem: $$[\Pi_{\mathrm{BJ}}]_{mn} = \sum_{k=0}^{n} \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{m-n} d_{mn}^{k\ell}\ \Phi^{k}_{(m-n-\ell)/2,\ell/2}$$

- efficient recursion of matrix elements
- low-rank matrix: efficient approximations

Applications



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$\mathbf{s}$-parametrized phase spaces $F_\rho(\Omega, \mathbf{s})$ as expectation values

$$F_\rho(\Omega, \mathbf{s}) := \mathrm{Tr}\,[\,\rho\,\mathcal{R}(\Omega) M_{\mathbf{s}} \mathcal{R}^\dagger(\Omega)]$$

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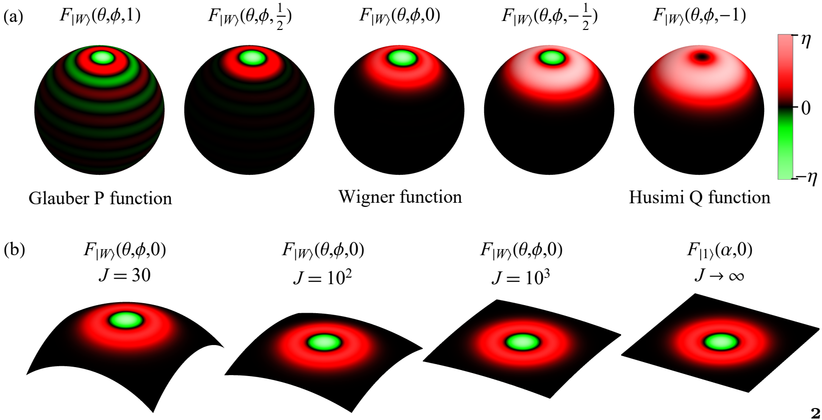
- of parity operators $M_{\mathbf{s}}$ ($d \times d$, diagonal matrices)
- SU(2) rotation operator $\mathcal{R}(\Omega) := e^{i\phi\mathcal{J}_z} e^{i\theta\mathcal{J}_y}$
- phase-space coordinate $\Omega := (\theta, \phi)$ via Euler angles
- tomography based on parity operators $M_{\mathbf{s}}$

Spin as a collection of qubits

- W state of $N = 2J$ symmetric qubits $|W\rangle$ (Dicke state)
- s-parametrized phase-space function $F_{|W\rangle}(\theta, \phi, s)$
- convergence to quantum optics as $N \to \infty$



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time evolution of the density operator ρ

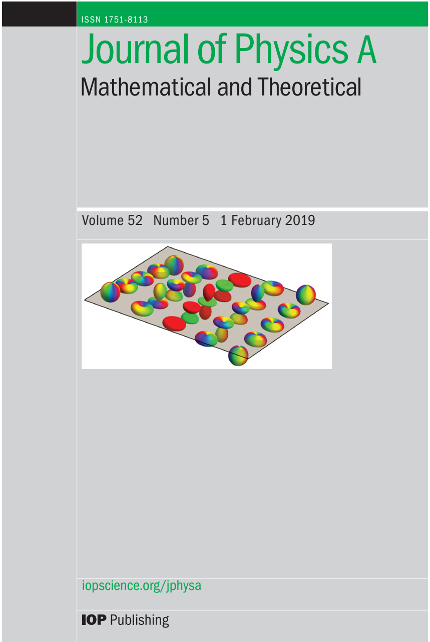
$$i\frac{\partial \rho}{\partial t} = [\mathcal{H}, \rho] = \mathcal{H}\rho - \rho\mathcal{H}$$

Moyal equation: star commutator $[W_{\mathcal{H}}, W_\rho]_\star$ in phase space

$$i\frac{\partial W_\rho}{\partial t} = [W_{\mathcal{H}}, W_\rho]_\star := W_{\mathcal{H}} \star W_\rho - W_\rho \star W_{\mathcal{H}}.$$

star products satisfy $W_{AB} = W_A \star W_B$

- generalised previous qubit results to qudits
- spin-weighted spherical harmonics: Newman and Penrose for general relativity
- exact and approximate time evolutions for qudits
- efficient approximations of spin phase spaces



Result: s-parametrized star products

- complicated form in general for arbitrary $\mathbf{s}$
- spin weight raising ð and lowering $\bar{ð}$ operators

$$f \star^{(\mathbf{s})} g = \sum_{\underline{a},\underline{b},\underline{c},\underline{d}} \lambda^{(\mathbf{s})}_{\underline{a},\underline{b},\underline{c},\underline{d}} [\dots (\bar{ð})^{a_2} (ð)^{b_1} (\bar{ð})^{a_1} f][\dots (\bar{ð})^{d_2} (ð)^{c_1} (\bar{ð})^{d_1} g]$$

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efficient approximation which recovers quantum optics

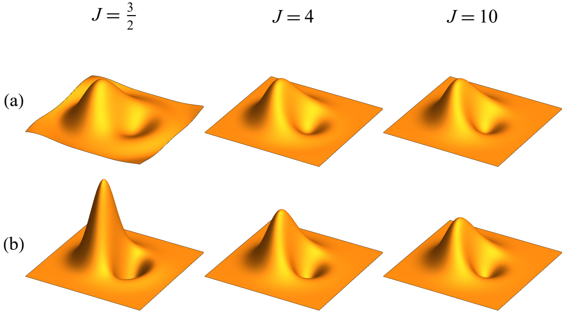
$$f \star^{(\mathbf{s})} g = f \exp[\tfrac{(1-\mathbf{s})}{2} \overleftarrow{\partial}_{\alpha} \overrightarrow{\partial}_{\alpha^*} - \tfrac{(1+\mathbf{s})}{2} \overleftarrow{\partial}_{\alpha^*} \overrightarrow{\partial}_{\alpha}] g + \mathcal{O}(J^{-1}).$$

Approximation of Wigner functions

- time evolution via the Moyal equation
- WF: classical Poisson bracket + quantum corrections
- efficient approximations for large spins

(a) exact

(b) approximate



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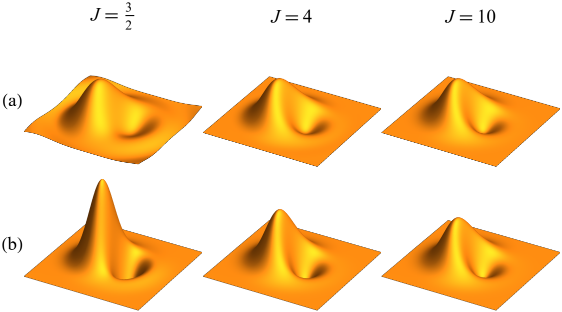

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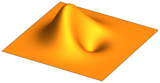
Convergence to quantum optics for $J \to \infty$



Acknowledgments

- University of Oxford:

  Prof. Simon Benjamin, Suguru Endo

- Technical University of Munich:

  Prof. Steffen Glaser, Robert Zeier, Frederik vom Ende

- University of Vienna: Prof. Maurice de Gosson

References

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