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Gradient magnetometry with entangled atomic ensembles

- Iagoba Apellaniz -

Dept. of Theoretical Physics
University of the Basque Country (EHU/UPV)
Bilbao, Spain.

September 28, 2018

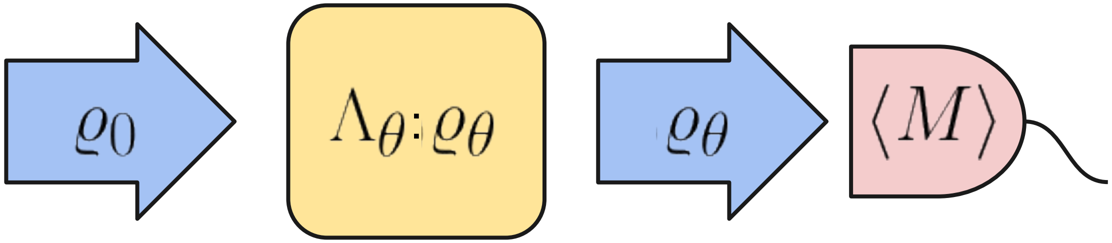
Budapest

#EntanglementDays18

Introduction

- Quantum metrology studies **precision bounds** when quantum systems are used for the estimation process.

Ex.: Single parameter estimation process

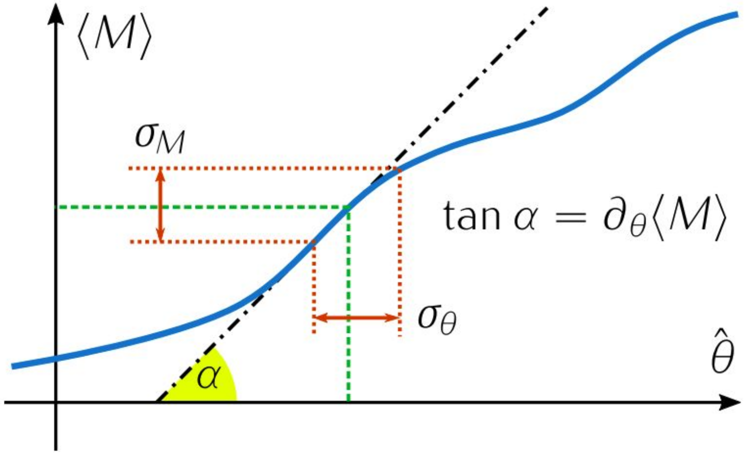


Precision bounds

$$\Lambda_\theta : \varrho_0 \equiv e^{-i\theta J_z} \varrho_0 e^{+i\theta J_z} \qquad J_z := \sum_{n=1}^{N} j_z^{(n)}$$

Error propagation formula:

$$(\Delta\theta)^{-2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2}$$



Cramér-Rao inequality - Quantum Fisher information:

$$(\Delta\theta)^{-2} \leqslant \mathcal{F}_Q[\varrho, J_z]$$

$$\mathcal{F}_Q[\varrho, J_z] = 2\sum_{\lambda,\mu} \frac{(p_\lambda - p_\mu)^2}{p_\lambda + p_\mu} |\langle \lambda | J_z | \mu \rangle|^2$$

Pure states,

$$\mathcal{F}_Q[|\psi\rangle, J_z] = 4(\Delta J_z)^2$$

Entanglement and QFI

The QFI is convex over the states. Hence, we search for the **best separable state** among pure states only

$$\mathcal{F}_{\mathrm{Q}}[p\varrho_1+(1-p)\varrho_2] \leqslant p\mathcal{F}_{\mathrm{Q}}[\varrho_1]+(1-p)\mathcal{F}_{\mathrm{Q}}[\varrho_2]$$

The precision bound scales with *N* as

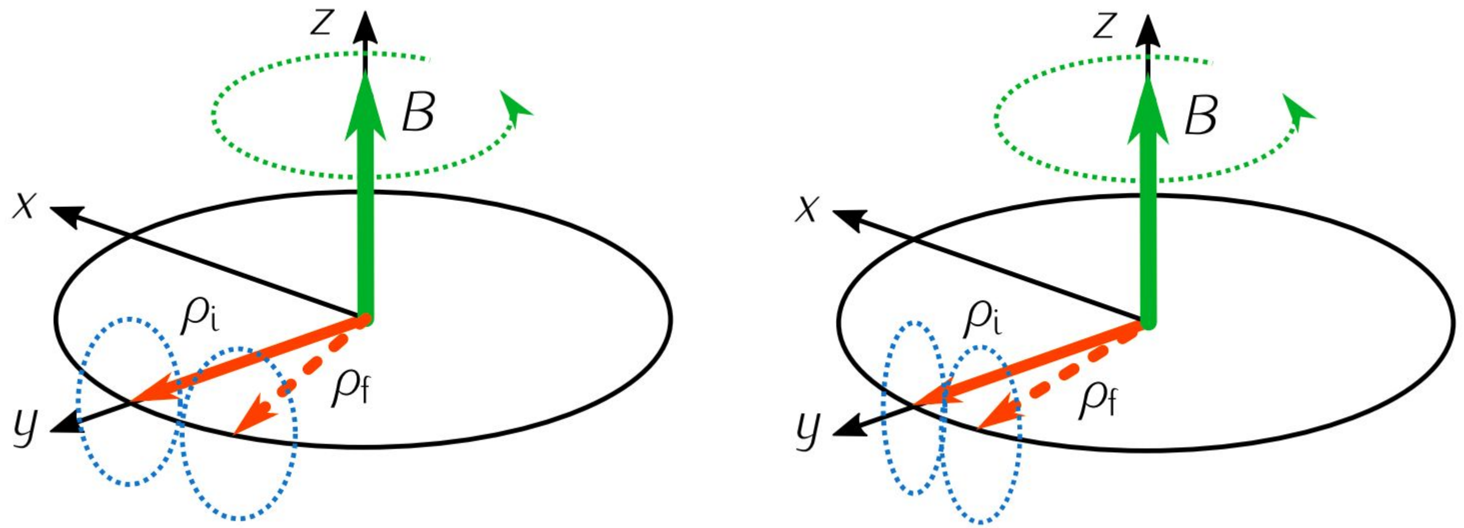
$$(\Delta\theta)^2 \sim \frac{1}{N}$$ *Shot-noise scaling*

whereas the best pure state can reach to

$$(\Delta\theta)^2 \sim \frac{1}{N^2}$$ *Heisenberg scaling*

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle, J_z] = 4(\Delta J_z)^2 \qquad J_z := \sum_{n=1}^{N} j_z^{(n)}$$

Spin-squeezed state



$$\xi_s^2 := N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2} \geqslant 1$$

A. Sørensen et al. Nature **409**, 63 (2001)

$$\mathcal{F}_Q[|\psi\rangle, J_z] = 4(\Delta J_z)^2$$

$$J_z := \sum_{n=1}^{N} j_z^{(n)}$$

Greenberger–Horne–Zeilinger state

$$|\mathrm{GHZ}\rangle = \frac{|0\ldots00\rangle + |1\ldots11\rangle}{\sqrt{2}}$$

- The GHZ state is a maximally entangled state.
- The GHZ maximises the variance of the *Jz* angular momentum as

$$(\Delta\theta)^{-2} = \mathcal{F}_{\mathrm{Q}}[|\mathrm{GHZ}\rangle, J_z] = 4(\Delta J_z)^2 = N^2.$$

- Hence, it saturates the achievable precision for a quantum system.

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle, J_z] = 4(\Delta J_z)^2 \qquad J_z := \sum_{n=1}^{N} j_z^{(n)}$$

Multiparameter estimation

Let us introduce the precision bounds for the **multiparametric case**, when the state encodes all the wanted parameters , $\Lambda_{\theta_1,\theta_2,\ldots,\theta_k} : \varrho_0$.

- Cramér-Rao matrix inequality: (Unitary evolution, G_i *are the generators)*

$$\mathrm{Cov}(\theta_i, \theta_j) \geqslant [\boldsymbol{\mathcal{F}}_{\mathrm{Q}}^{-1}]_{i,j}$$

$$[\boldsymbol{\mathcal{F}}_{\mathrm{Q}}]_{i,j} := \mathcal{F}_{\mathrm{Q}}[\rho, G_i, G_j]$$

- The saturability is guaranteed when a **simultaneous optimal measurement** is possible for all the parameters. In other words,

$$[L_i, L_j] = 0$$

where L_i is the symmetric logarithmic derivative.

M. G. A. Paris, Int. J. Quant. Inf. 7, 125 (2009)

Motivation

- There are interesting **entangled states insensitive to the homogeneous** fields, such as the singlet state.

Ex.: *G. Tóth et al. New J. Phys. 12 053007 (2010)*

- Gradient magnetometry is one of the simplests **multiparameter** estimation problems.

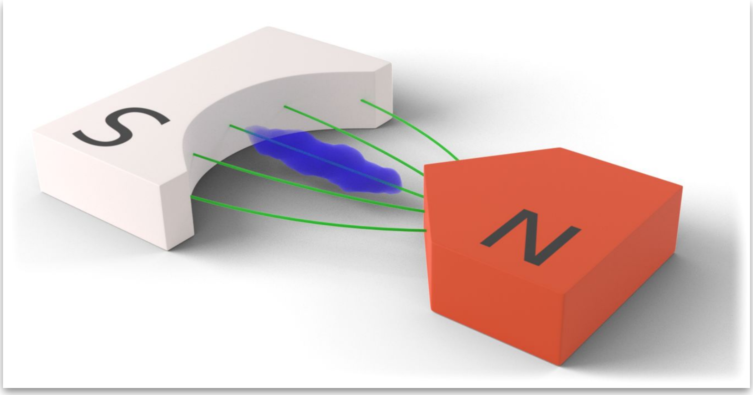
- There are **many experiments** that study gradient magnetometry from the quantum metrology perspective.

Gradient Magnetometry

- We study achievable precisions of the estimation of the **MAGNETIC FIELD GRADIENT**.

- We study **PRECISION BOUNDS** for:

1. Spin-chains
2. Double well experiments
3. Single cloud of atoms



Related work: *S. Altenburg et al. Phys. Rev. A* ***96****, 042319 (2017)*

$$\boldsymbol{B}(x,0,0) = \boldsymbol{B}_0 + x\boldsymbol{B}_1 + \mathcal{O}(x^2)$$

Homogeneous part ($\boldsymbol{B}_0$) — **Gradient parameter** ($\boldsymbol{B}_1$)

The *N*-particle state and the setup

Factorizable between the spatial and spin parts:

$$\varrho = \varrho^{(\mathrm{x})} \otimes \varrho^{(\mathrm{s})}$$

Spatial part:

$$\varrho^{(\mathrm{x})} = \int \frac{P(\boldsymbol{x})}{\langle \boldsymbol{x}|\boldsymbol{x}\rangle} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \, \mathrm{d}\boldsymbol{x}$$

Spin chains, cold atomic ensembles, double well experiments, …

- Each particle interacts with the magnetic field:

$$h^{(n)} = \gamma B_z^{(n)} \otimes j_z^{(n)}$$

- The gradient parameter B_1 is encoded in the phase b_1:

$$U = e^{-i(b_0 H_0 + b_1 H_1)}$$

- Generators of phase-shifts:

$$H_0 = \sum_{n=1}^{N} j_z^{(n)} = J_z$$

$$H_1 = \sum_{n=1}^{N} x^{(n)} j_z^{(n)}$$

Cramér-Rao Precision Bounds

- Fos states INSENSITIVE to the homogeneous fields:

$$(\Delta b_1)^{-2}|_{\max} = \mathcal{F}_{\mathrm{Q}}[\varrho, H_1, H_1] = \mathcal{F}_{\mathrm{Q}}[\varrho, H_1]$$

- Fos states SENSITIVE to the homogeneous fields: $\mathcal{F}_{ij} := \mathcal{F}_{\mathrm{Q}}[\varrho, H_i, H_j]$

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_{11} - \frac{\mathcal{F}_{01}\mathcal{F}_{10}}{\mathcal{F}_{00}}$$

Quantum Fisher Information (QFI)

$$\mathcal{F}_{\mathrm{Q}}[\varrho, A, B] := 2\sum_{k,k'} \frac{(p_k - p_{k'})^2}{p_k + p_{k'}} A_{k,k'} B_{k',k}$$

States insensitive to homogeneous fields (H_0)

The matrix elements of H_1

$$(H_1)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu} = \delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N} x_n j^{(n)}|\nu\rangle$$

Precision bound for the **gradient** parameter

$$(\Delta b_1)^{-2}|_{\max} = \sum_{n,m}^{N}\int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, j_z^{(m)}]$$

Note: *This bound is **invariant** under spatial translations.*

States sensitive to homogeneous fields (H_0)

The matrix elements of H_0

$$(H_0)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu} = \delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N} j_z^{(n)}|\nu\rangle$$

Precision bound for the **gradient** parameter

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m}^{N} \int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, j_z^{(m)}] - \frac{\left(\sum_{n=1}^{N} \int x_n P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, J_z]\right)^2}{\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, J_z]}$$

Note: *This bound is also **invariant** under spatial translations.*

Simultaneous measurements

Condition for simultaneous measurements:

$$[L(\varrho, H_0), L(\varrho, H_1)] = 0$$

Symmetric logarithmic derivative (SLD):

$$L(\varrho, H_0) = \mathbf{1}^{(\mathrm{x})} \otimes L(\varrho^{(s)}, J_z)$$

$$L(\varrho, H_1) = \sum_{n=1}^{N} \int \mathrm{d}\boldsymbol{x}\, x_n |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \otimes L(\varrho^{(s)}, j_z^{(n)})$$

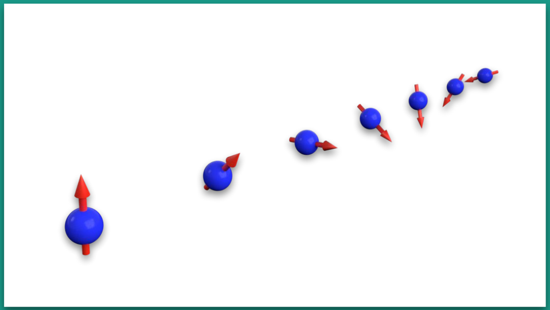
Example: PI states

$$L(\varrho, H_1) = \hat{\mu}^{(\mathrm{x})} \otimes L(\varrho^{(s)}, J_z)$$

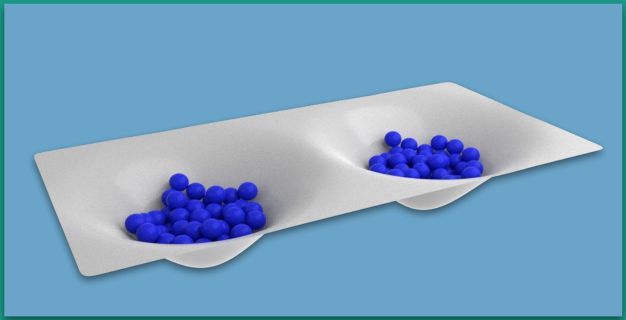
See Phys. Rev. A 97, 053603 (2018), for more details.

Spin Chain and Double Well

$$P(\boldsymbol{x}) = \prod_{n=1}^{N} \delta(x_n - na)$$



$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n + a) \prod_{n=N/2+1}^{N} \delta(x_n - a)$$



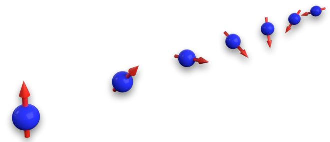
Gradient magnetometry with spin chains

State totally polarized in the y direction: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$

PRECISION BOUND:

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_{11} - \frac{\mathcal{F}_{01}\mathcal{F}_{10}}{\mathcal{F}_{00}} \qquad \sigma_{\rm ch}^2 = a^2 \frac{N^2-1}{12}$$

$$(\Delta b_1)^{-2}|_{\max} = 2\sigma_{\rm ch}^2 N j$$



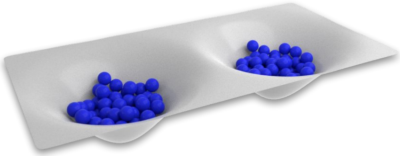
Double wells for gradient estimation

The variance for double wells: $\sigma_{\rm dw}^2 = a^2$

The **entangled** state that saturates the *Heisenberg limit*:

$$|\psi\rangle = \frac{|j\cdots j\rangle^{(\rm L)}|-j\cdots -j\rangle^{(\rm R)} + |-j\cdots -j\rangle^{(\rm L)}|j\cdots j\rangle^{(\rm R)}}{\sqrt{2}}$$

$$(\Delta b_1)^{-2}|_{\max} = 4\sigma_{\rm dw}^2 N^2 j^2$$



Double wells for gradient estimation

The variance for double wells: $\sigma_{\mathrm{dw}}^2 = a^2$

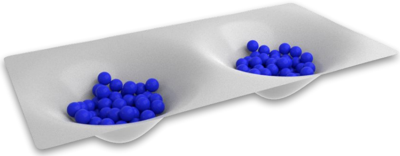
The **entangled** state that saturates the *Heisenberg limit*:

$$|\psi\rangle = \frac{|j\cdots j\rangle^{(\mathrm{L})}|-j\cdots -j\rangle^{(\mathrm{R})} + |-j\cdots -j\rangle^{(\mathrm{L})}|j\cdots j\rangle^{(\mathrm{R})}}{\sqrt{2}}$$

$$(\Delta b_1)^{-2}|_{\max} = 4\sigma_{\mathrm{dw}}^2 N^2 j^2$$

For product states: $\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}|\psi\rangle^{(\mathrm{R})}, H_1] = 2a^2 \mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}, J_z^{(\mathrm{L})}]$

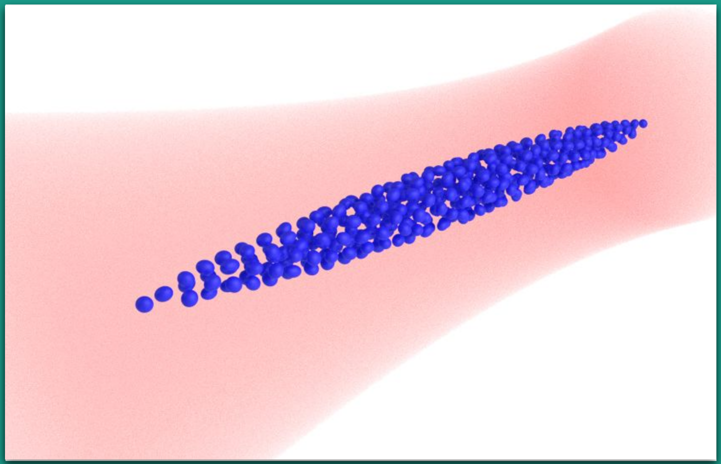
States	$\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{L})}, J_z^{(\mathrm{L})}]$	$(\Delta b_1)^{-2}|_{\max}$
$|j\rangle_y^{\otimes N_{\mathrm{L}}} \otimes |j\rangle_y^{\otimes N_{\mathrm{L}}}$	$2N_{\mathrm{L}}j$	$2a^2Nj$
$|\Psi_{\mathrm{sep}}\rangle \otimes |\Psi_{\mathrm{sep}}\rangle$	$4N_{\mathrm{L}}j^2$	$4a^2Nj^2$
$|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle$	N_{L}^2	$a^2N^2/2$
$|\mathrm{D}_{N_{\mathrm{L}}}\rangle_x \otimes |\mathrm{D}_{N_{\mathrm{L}}}\rangle_x$	$N_{\mathrm{L}}(N_{\mathrm{L}}+2)/2$	$a^2N(N+4)/4$



One-Dimensional Ensemble of Atoms

We now show bounds for an arbitrary **PERMUTATIONALLY INVARIANT** (PI) probability distribution function.

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k} \Pi_k [P(\boldsymbol{x})]$$



Precision bounds for a single ensemble

For states **INSENSITIVE** to the homogeneous fields:

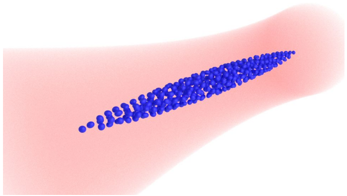
$$(\Delta b_1)^{-2}|_{\max} = (\sigma^2 - \eta)\sum_{n=1}^{N} \mathcal{F}_Q[\varrho^{(s)}, j_z^{(n)}]$$

For states **SENSITIVE** to the homogeneous fields:

$$(\Delta b_1)^{-2}|_{\max} = (\sigma^2 - \eta)\sum_{n=1}^{N} \mathcal{F}_Q[\varrho^{(s)}, j_z^{(n)}] + \eta \mathcal{F}_Q[\varrho^{(s)}, J_z]$$

Correlation between particle positions:

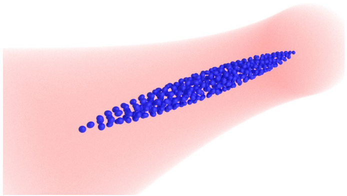
$$\frac{-\sigma^2}{N-1} \leqslant \eta \leqslant \sigma^2$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$
- Singlet: $\varrho_{\rm singlet}^{(\rm s)} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$
- Best separable: $|\psi_{\rm sep}\rangle = \left(\frac{|-j\rangle + |+j\rangle}{\sqrt{2}}\right)^{\otimes N}$
- $|{\rm GHZ}\rangle = \frac{|00\cdots 00\rangle + |11\cdots 11\rangle}{\sqrt{2}}$
- Unpolarized Dicke state (*x* and *z*):

$$|{\rm D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k (|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm tp}|_{\max} = 2\sigma^2 N j$$

- Best separable: $|\psi_{\rm sep}\rangle = \left(\frac{|-j\rangle + |+j\rangle}{\sqrt{2}}\right)^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm sep}|_{\max} = 4\sigma^2 N j^2$$

- Unpolarized Dicke state (*x* and *z*):

$$|{\rm D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k(|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$

$$(\Delta b_1)^{-2}_{\rm D}|_{\max} = (\sigma^2 - \eta)N$$

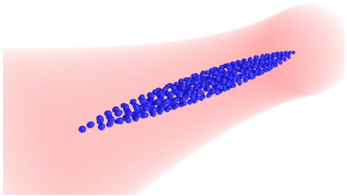
$$(\Delta b_1)^{-2}_{{\rm D},x}|_{\max} = (\sigma^2 - \eta)N + \eta\frac{N(N+2)}{2}$$

- Singlet: $\varrho^{(\rm s)}_{\rm singlet} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$

$$(\Delta b_1)^{-2}_{\rm singlet}|_{\max} = (\sigma^2 - \eta)N\frac{4j(j+1)}{3}$$

- $|{\rm GHZ}\rangle = \frac{|00\cdots00\rangle + |11\cdots11\rangle}{\sqrt{2}}$

$$(\Delta b_1)^{-2}_{\rm GHZ}|_{\max} = (\sigma^2 - \eta)N + \eta N^2$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm tp}|_{\max} = 2\sigma^2 N j$$

- Best separable: $|\psi_{\rm sep}\rangle = \left(\frac{|-j\rangle + |+j\rangle}{\sqrt{2}}\right)^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm sep}|_{\max} = 4\sigma^2 N j^2$$

- Unpolarized Dicke state (*x* and *z*):

$$|{\rm D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k(|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$

$$(\Delta b_1)^{-2}_{\rm D}|_{\max} = (\sigma^2 - \eta) N$$

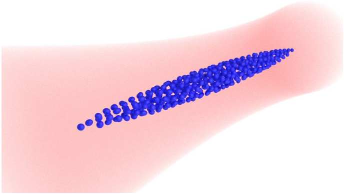
$$(\Delta b_1)^{-2}_{{\rm D},x}|_{\max} = (\sigma^2 - \eta) N + \eta \frac{N(N+2)}{2}$$

- Singlet: $\varrho^{(\rm s)}_{\rm singlet} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$

$$(\Delta b_1)^{-2}_{\rm singlet}|_{\max} = (\sigma^2 - \eta) N \frac{4j(j+1)}{3}$$

- $|{\rm GHZ}\rangle = \frac{|00\cdots 00\rangle + |11\cdots 11\rangle}{\sqrt{2}}$

$$(\Delta b_1)^{-2}_{\rm GHZ}|_{\max} = (\sigma^2 - \eta) N + \eta N^2$$



Conclusions

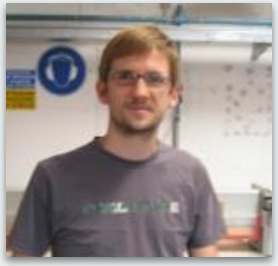
- We obtained **GENERAL FORMULAS** to compute the precision bounds for gradient magnetometry for spin-chains, double-wells, atomic single clouds and BECs.
- These bounds are based on the **INTERNAL STATE** of the system.
- Among the bounds we presented for an atomic cloud, there is the bound for the **BEST SEPARABLE STATE**.
- We found that all bounds appearing on this work are **SATURABLE**.

Thank you for your attention!

Please, for more information, visit our preprint at Phys. Rev. A **97**, 053603 (2018)


Iñigo Urizar-Lanz


Zoltán Zimborás


Philipp Hyllus


Géza Tóth