Symmetry Principles in Quantum Simulation

with Applications to the Control of Closed and Open Systems

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Symmetry guidelines for answering:

1. when is a quantum hardware universal?
   interplay of controls and coupling architecture
2. when can quantum system *A* simulate system *B* ?
   in particular: least state-space overhead
3. what are the reachable sets under collective controls?

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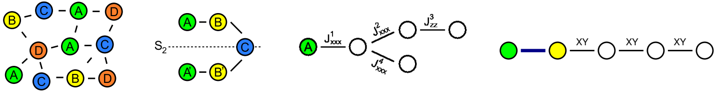
More generally:
what are the reachable sets in open systems?

Hamiltonian components: $H_{\text{tot}} = H_0 + \sum_j u_j H_j$ in $\dot{\rho} = -i[H_{\text{tot}}, \rho]$

- vertices: controls = pulses (type-wise joint local actions)
- edges: drift = couplings (Ising-ZZ; Heisenberg-XX, XY, XXX)
- system algebra $\mathfrak{k} := \langle iH_0, iH_j \,|\, j = 1, 2, \dots, m \rangle_{\text{Lie}}$



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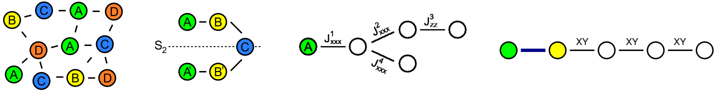


- Consider closed control system with $\mathfrak{k} = \langle iH_\nu \rangle_{Lie}$

Definition

The *symmetry* of the Hamiltonians $\{iH_\nu\}$ is expressed by the *centraliser* (or *commutant*) of $\mathfrak{k}$ in $\mathfrak{su}(N)$

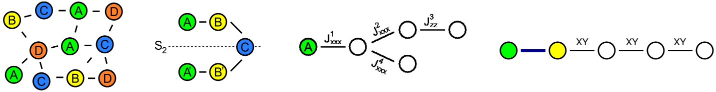
$$\mathfrak{k}' := \{s \in \mathfrak{su}(N) | [s, H_\nu] = 0 \quad \forall \nu = d; 1, 2, \ldots, m\} .$$

It collects all *constants of motion* under $\mathbf{K} = \langle \exp \mathfrak{k} \rangle$.

Control system Σ with algebra $\mathfrak{k} = \langle iH_\nu \,|\, \nu = d; 1, 2, \dots, m \rangle_{\text{Lie}}$.

1. coupling graph to H_d connected
2. no symmetry ($\mathfrak{k}'$ trivial)

(1) and (2) $\Rightarrow$ dynamic algebra $\mathfrak{k}$ is *simple*



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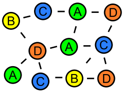

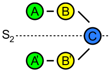

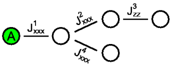

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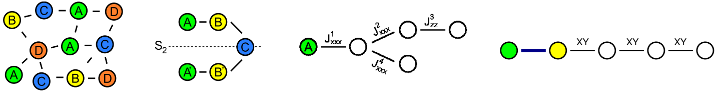

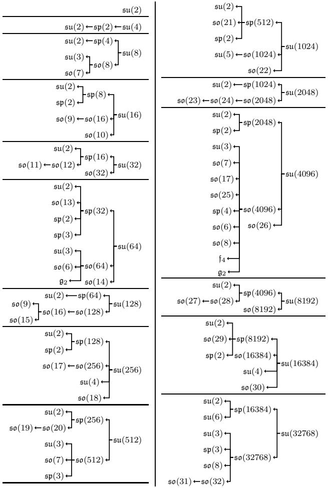

su(2)
su(2) ← sp(2) ← su(4)
su(2) ← sp(4)
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so(9) ← so(16)
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so(11) ← so(12)
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su(64)
su(3)
so(6)
so(64)
g2
so(14)
su(2) ← sp(64)
so(9)
su(128)
so(16) ← so(128)
so(15)
su(2)
sp(128)
sp(2)
so(17) ← so(256)
su(256)
su(4)
so(18)
su(2)
sp(256)
so(19) ← so(20)
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so(27) ← so(28)
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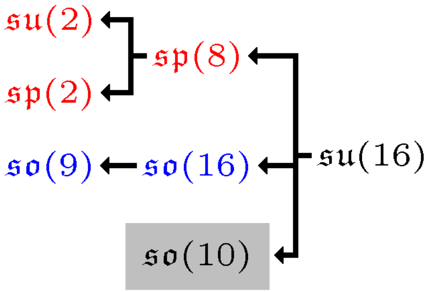

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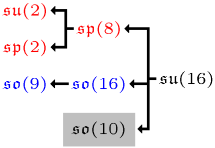
3. itself neither of orthogonal nor symplectic type



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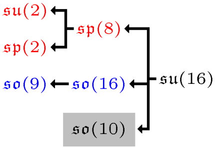

Algorithm : Check for *conjugation* to $\mathfrak{so}(N)$ or $\mathfrak{sp}(\frac{N}{2})$ for *n*-qubit drift and control Hamiltonians $\{iH_d; H_1, \dots, H_m\}$

1. For each Hamiltonian $H_\nu \in \{H_d; H_1, \dots, H_m\}$ determine all non-singular *solutions to the homogeneous linear eqn.*

   $$\mathcal{S}_\nu := \{S \in SL(N) | SH_\nu + H_\nu^t S = 0\} \,\widehat{=}\, \ker\left(H_\nu \otimes \mathbb{1} + \mathbb{1} \otimes H_\nu\right)$$

2. Check intersection of all sets of solutions

   $$\mathcal{S} = \bigcap_\nu \mathcal{S}_\nu.$$

   if $S\bar{S} = +\mathbb{1}$: $\mathfrak{k} \subseteq \mathfrak{so}(N)$

   if $S\bar{S} = -\mathbb{1}$: $\mathfrak{k} \subseteq \mathfrak{sp}(\frac{N}{2})$

   if $\mathcal{S} = \{\}$: $\mathfrak{k}$ of other type

Complexity $\mathcal{O}(N^6)$, as in Liouville space N^2 equations have to be solved by *LU* decomposition.

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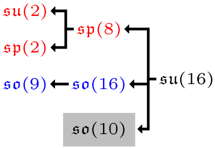
$\mathfrak{k}$ must include max. orthogonal and symplectic types



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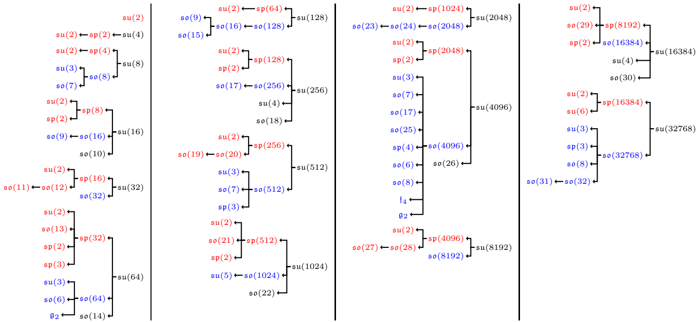

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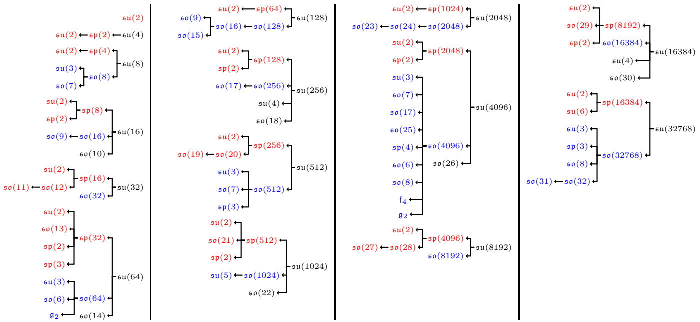

Theorem

Let $\{H_\nu \,|\, \nu = d; 1, 2, \dots, m\}$ *be drift and control Hamiltonians of control system* Σ *with system algebra* $\mathfrak{k}$.

Define $\Phi_{AB} := \{(iH_\nu \otimes \mathbb{1}_A + \mathbb{1}_B \otimes iH_\nu) \,|\, \nu = d, 1, \dots, m\}$.

Then Σ *is fully controllable, i.e.* $\mathfrak{k} = \mathfrak{su}(2^n)$, *iff*

- *joint commutant to* Φ_{AB} *is two-dimensional*

  i.e. $\Phi'_{AB} = \{\lambda \mathbb{1}, \mathrm{SWAP}_{AB}\}$.

  $[\Phi_{AB}] = [\text{symmetric}]_{\text{bosonic}} \oplus [\text{anti-symmetric}]_{\text{fermionic}}$

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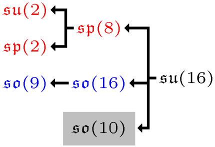

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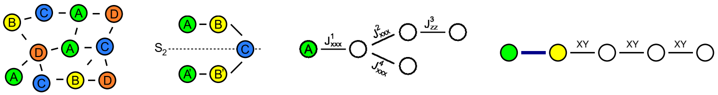

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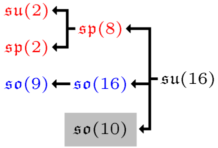

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Let Σ_A, Σ_B be control systems with irreducible system algebras $\mathfrak{k}_A, \mathfrak{k}_B$ over a given Hilbert space $\mathcal{H}$. Then

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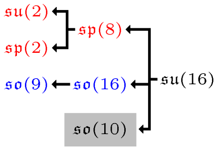

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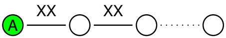	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
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for $n \bmod 4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
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NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

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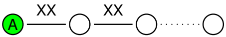
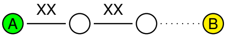
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
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I. Q-Control
II. Q-Simulation
Efficient Sim.
Collective Controls
III. Algorithms
IV. Applications
Outlook
Conclusions
A1
A2
A3

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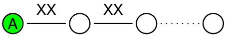
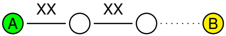
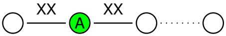
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
				
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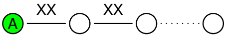
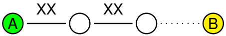
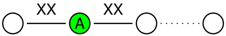
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
				
for n mod $4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for n mod $4 \in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$
	n	up to n	up to n	$\mathfrak{su}(2^n)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

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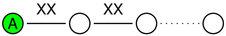
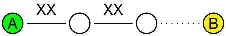
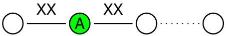
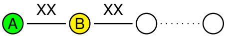
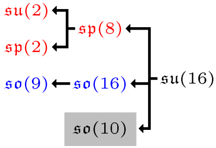
system type		fermionic	bosonic	system alg.
n-spins-$\frac{1}{2}$	no. of levels	——— order of coupling ———		
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
				
for n mod 4 $\in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for n mod 4 $\in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$
	n	up to n	up to n	$\mathfrak{su}(2^n)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

system type n-spins-$\frac{1}{2}$	no. of levels	fermionic	bosonic	system alg.
		——— order of coupling ———		
(A) —XX— ( ) —XX— ( ) ······· ( )	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
(A) —XX— ( ) —XX— ( ) ······· (B)	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
( ) —XX— (A) —XX— ( ) ······· ( )				
for n mod $4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for n mod $4 \in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$
(A) —XX— (B) —XX— ( ) ······· ( )	n	up to n	up to n	$\mathfrak{su}(2^n)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).



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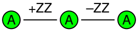
system type $n = 2k + 1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
	—”—	—”—	—”—
	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—

system type $n = 2k+1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A +ZZ A −ZZ A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A +ZZ B −ZZ A	—”—	—”—	—”—
	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—

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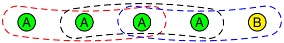
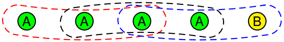
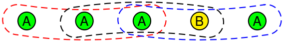
system type $n = 2k+1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A —(+ZZ)— A —(−ZZ)— A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A —(+ZZ)— B —(−ZZ)— A	—”—	—”—	—”—
A —(+ZZ)— A —(+ZZ)— A —(−ZZ)— A —(−ZZ)— A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—

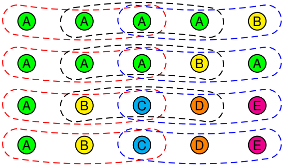
system type $n = 2k + 1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A $\overset{+ZZ}{—}$ A $\overset{-ZZ}{—}$ A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A $\overset{+ZZ}{—}$ B $\overset{-ZZ}{—}$ A	—"—	—"—	—"—
A $\overset{+ZZ}{—}$ A $\overset{+ZZ}{—}$ A $\overset{-ZZ}{—}$ A $\overset{-ZZ}{—}$ A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A $\overset{+ZZ}{—}$ A $\overset{+ZZ}{—}$ B $\overset{-ZZ}{—}$ A $\overset{-ZZ}{—}$ A	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—

system type $n = 2k + 1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A —+ZZ— A —−ZZ— A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A —+ZZ— B —−ZZ— A	—"—	—"—	—"—
A —+ZZ— A —+ZZ— A —−ZZ— A —−ZZ— A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A —+ZZ— A —+ZZ— B —−ZZ— A —−ZZ— A	—"—	—"—	—"—
A —+ZZ— B —+ZZ— C —−ZZ— B —−ZZ— A	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—

system type $n = 2k+1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A $\overset{+ZZ}{—}$ A $\overset{-ZZ}{—}$ A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A $\overset{+ZZ}{—}$ B $\overset{-ZZ}{—}$ A	—"—	—"—	—"—
A $\overset{+ZZ}{—}$ A $\overset{+ZZ}{—}$ A $\overset{-ZZ}{—}$ A $\overset{-ZZ}{—}$ A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A $\overset{+ZZ}{—}$ A $\overset{+ZZ}{—}$ B $\overset{-ZZ}{—}$ A $\overset{-ZZ}{—}$ A	—"—	—"—	—"—
A $\overset{+ZZ}{—}$ B $\overset{+ZZ}{—}$ C $\overset{-ZZ}{—}$ B $\overset{-ZZ}{—}$ A	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—
	—"—	—"—	—"—

system type $n = 2k+1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A —+ZZ— A —−ZZ— A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A —+ZZ— B —−ZZ— A	—”—	—”—	—”—
A —+ZZ— A —+ZZ— A —−ZZ— A —−ZZ— A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A —+ZZ— A —+ZZ— B —−ZZ— A —−ZZ— A	—”—	—”—	—”—
A —+ZZ— B —+ZZ— C —−ZZ— B —−ZZ— A	—”—	—”—	—”—
 A A A A B	—”—	—”—	—”—
 A A A B A	—”—	—”—	—”—
	—”—	—”—	—”—
	—”—	—”—	—”—

system type $n = 2k + 1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A +ZZ A −ZZ A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A +ZZ B −ZZ A	—”—	—”—	—”—
A +ZZ A +ZZ A −ZZ A −ZZ A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A +ZZ A +ZZ B −ZZ A −ZZ A	—”—	—”—	—”—
A +ZZ B +ZZ C −ZZ B −ZZ A	—”—	—”—	—”—
A A A A B	—”—	—”—	—”—
A A A B A	—”—	—”—	—”—
A B C D E	—”—	—”—	—”—
	—”—	—”—	—”—

system type $n = 2k + 1$ spins-$\frac{1}{2}$	no. of levels	bosonic coupling order	system alg. $\mathfrak{sp}(2^{n-1})$
A —+ZZ— A —−ZZ— A	$n = 3$	up to $n = 3$	$\mathfrak{sp}(8/2)$
A —+ZZ— B —−ZZ— A	—”—	—”—	—”—
A —+ZZ— A —+ZZ— A —−ZZ— A —−ZZ— A	$n = 5$	up to $n = 5$	$\mathfrak{sp}(32/2)$
A —+ZZ— A —+ZZ— B —−ZZ— A —−ZZ— A	—”—	—”—	—”—
A —+ZZ— B —+ZZ— C —−ZZ— B —−ZZ— A	—”—	—”—	—”—
 A A A A B	—”—	—”—	—”—
A A A B A	—”—	—”—	—”—
A B C D E	—”—	—”—	—”—
A B C D E	—”—	—”—	—”—

Consider:

1. linear control system: $\dot{x}(t) = Ax(t) + Bv$
2. bilinear control system: $\dot{X}(t) = (A + \sum_j u_j B_j) X(t)$

Conditions for Full Controllability:

1. in linear systems: rank $[B, AB, A^2B, \ldots, A^{N-1}B] = N$
2. in bilinear systems: $\langle A, B_j \,|\, j = 1, 2, \ldots, m \rangle_{\text{Lie}} = \mathfrak{k} = \mathfrak{su}(N)$

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2. bilinear control system: $\dot{X}(t) = (A + \sum_j u_j B_j)X(t)$

   e.g.: Ham. quantum system $\dot{U}(t) = -i(H_d + \sum_j u_j H_j)U(t)$

   open quantum system $\dot{F}(t) = -(i\,\mathrm{ad}_{H_d} + i \sum_j u_j\,\mathrm{ad}_{H_j} + \Gamma_L)F(t)$

Conditions for Full Controllability:

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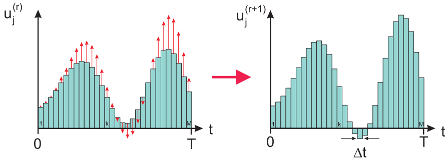
J. Magn. Reson. **172** (2005), 296 and *Phys. Rev. A* **72** (2005), 042331

concurrent (GRAPE)



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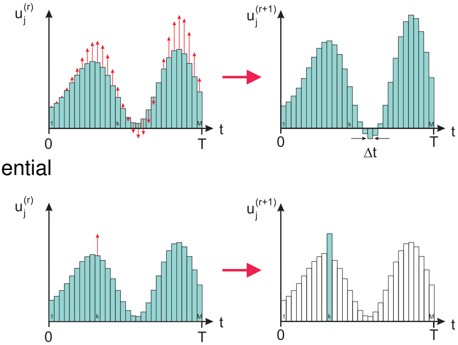
concurrent (GRAPE)

sequential



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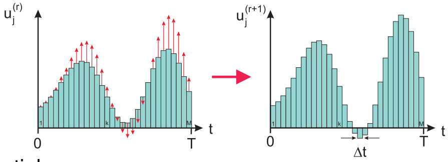
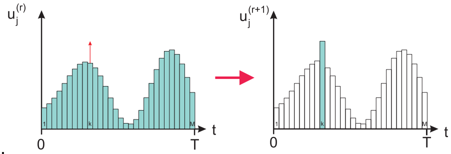
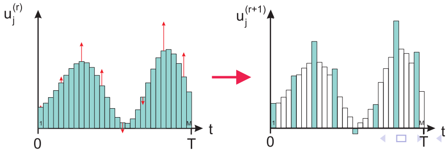


sequential



hybrid



0. initialise amplitudes $u_j^{(0)}(t_k) \in \mathcal{U} \subseteq \mathbb{R}$ for all times t_k with $k \in \mathcal{T}_k^{(0)} := \{1, 2, \ldots, M\}$, def. X_0, $X_{\text{tar}}^\dagger$.
1. exponentiate $X_k = e^{\Delta t A_u(t_k)}$ for all $k \in \mathcal{T}_k^{(r)}$ with $A_u(t_k) := A + \sum_j u_j(t_k) B_j$
2. multiplication I $\quad X_{k:0} := X_k \cdot X_{k-1} \cdots X_1 (\cdot X_0 = \mathbb{1})$
3. multiplication II $\quad \Lambda_{M+1:k+1}^\dagger := X_{\text{tar}}^\dagger \cdot X_M \cdot X_{M-1} \cdots X_{k+1}$
4. evaluate fidelity $f = \frac{1}{N} |\operatorname{tr}\{\Lambda_{M+1:k+1}^\dagger X_{k:0}\}|$
5. approximate gradients $\frac{\partial f(X(t_k))}{\partial u_j}$ for all $k \in \mathcal{T}_k^{(r)}$
6. update amplitudes for all $k \in \mathcal{T}_k^{(r)}$ e.g. $u_j^{(r+1)}(t_k) = u_j^{(r)}(t_k) + F(\alpha_k, \mathbf{Hess}_k^{-1}, \frac{\partial f(X(t_k))}{\partial u_j})$
7. loops
   inner: while $||\frac{\partial f_k}{\partial u_j}|| > g_{\text{limit}}$ for $k \in \mathcal{T}_k^{(r)}$ **goto step 1** ($s \mapsto s+1$)
   outer: else **goto step 1** with new set $\mathcal{T}_k^{(r+1)}$ ($r \mapsto r+1$)

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   e.g. $u_j^{(r+1)}(t_k) = u_j^{(r)}(t_k) + F(\alpha_k, \mathbf{Hess}_k^{-1}, \frac{\partial f(X(t_k))}{\partial u_j})$
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   with $k \in \mathcal{T}_k^{(0)} := \{1, 2, \ldots, M\}$, def. X_0, $X_{\rm tar}^\dagger$.
1. exponentiate $X_k = e^{\Delta t A_u(t_k)}$ for all $k \in \mathcal{T}_k^{(r)}$ with
   $A_u(t_k) := A + \sum_j u_j(t_k) B_j$
2. multiplication I $\quad X_{k:0} := X_k \cdot X_{k-1} \cdots X_1 (\cdot X_0 = \mathbb{1})$
3. multiplication II $\quad \Lambda_{M+1:k+1}^\dagger := X_{\rm tar}^\dagger \cdot X_M \cdot X_{M-1} \cdots X_{k+1}$
4. evaluate fidelity $f = \frac{1}{N} |\operatorname{tr}\{\Lambda_{M+1:k+1}^\dagger X_{k:0}\}|$
5. approximate gradients $\frac{\partial f(X(t_k))}{\partial u_j}$ for all $k \in \mathcal{T}_k^{(r)}$
6. update amplitudes for all $k \in \mathcal{T}_k^{(r)}$
   e.g. $u_j^{(r+1)}(t_k) = u_j^{(r)}(t_k) + F(\alpha_k, \textbf{Hess}_k^{-1}, \frac{\partial f(X(t_k))}{\partial u_j})$
7. loops
   inner: while $||\frac{\partial f_k}{\partial u_j}|| > g_{\rm limit}$ for $k \in \mathcal{T}_k^{(r)}$ **goto step 1** $(s \mapsto s+1)$
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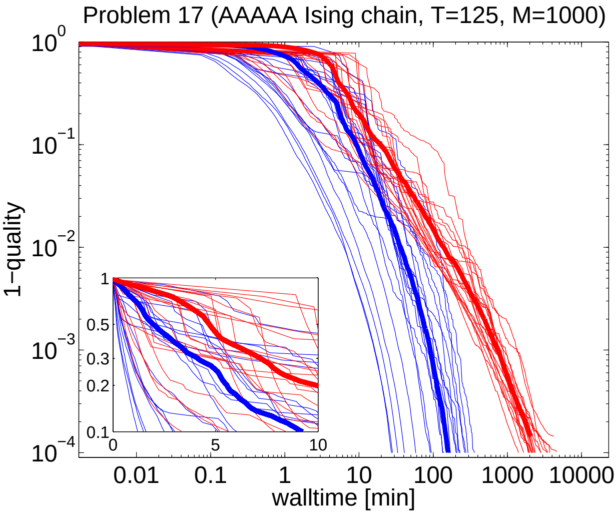
Comparison: 5-Qubit QFT

Ising Chain with Stark Shift

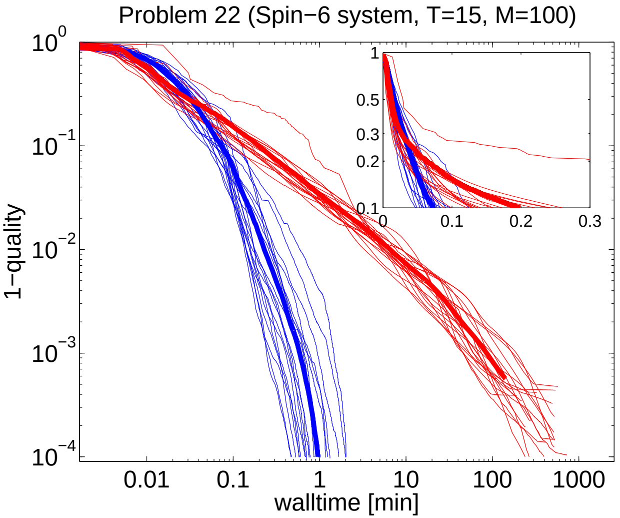
PRA **84** 022305 (2011)

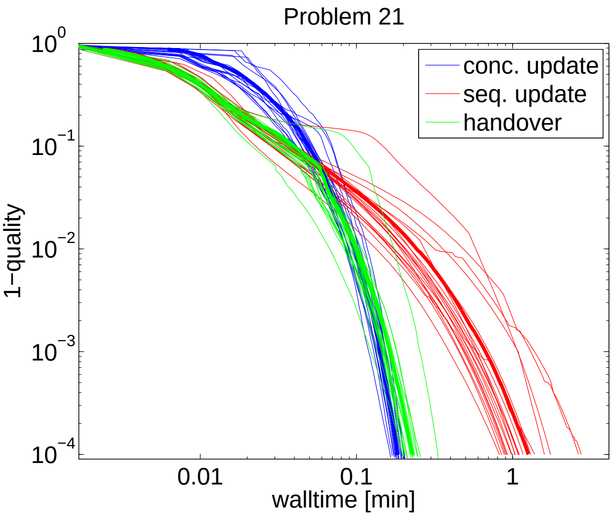


concurrent (GRAPE) sequential (KROTOV)

Comparison: Random Unitary

Driven Spin-6 System: model of Seth M. and Frank W.



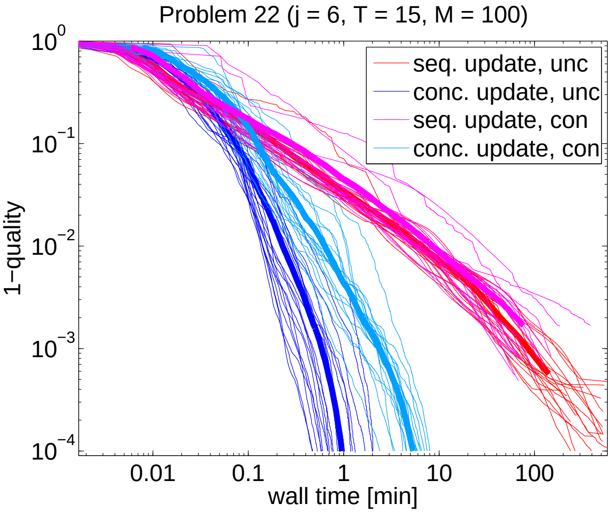


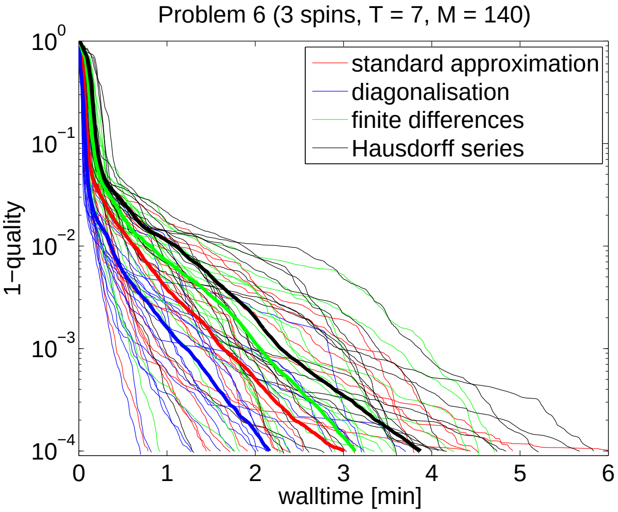
Problem 21
1−quality
walltime [min]
10^0
10^−1
10^−2
10^−3
10^−4
0.01
0.1
1
conc. update
seq. update
handover

Comparison: Random Unitary

Driven Spin-6 System

PRA **84** 022305 (2011)

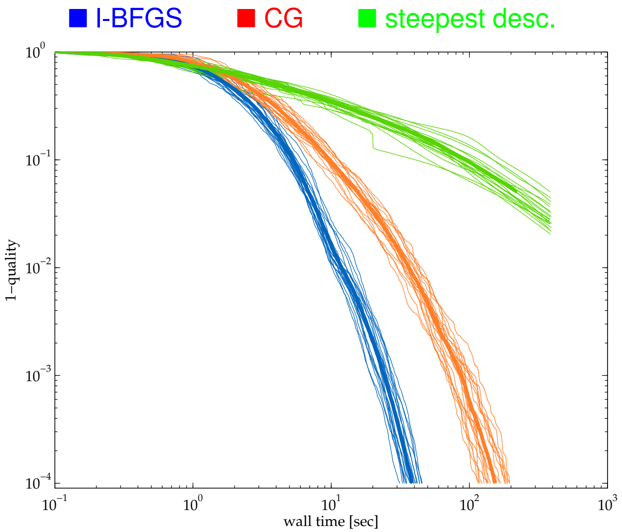




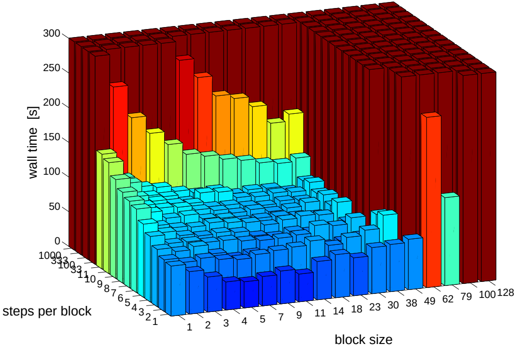
Problem 6 (3 spins, T = 7, M = 140)
standard approximation
diagonalisation
finite differences
Hausdorff series
1−quality
walltime [min]
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
1
2
3
4
5
6

Comparison to Conjugate Gradients

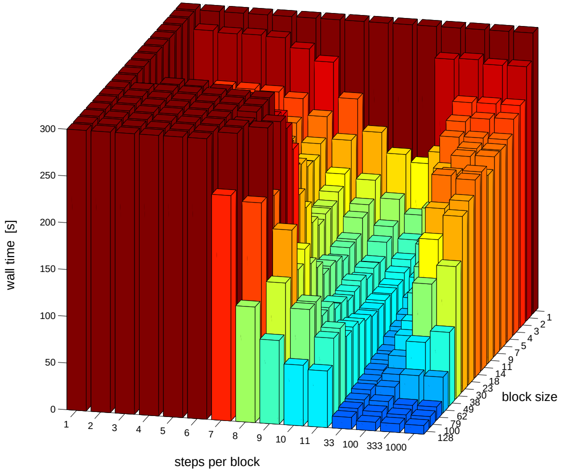
Driven Spin-3 System: Rand. Unitary   Uwe Sander's PhD Thesis (2010)



- first-order hybrids



- second-order hybrids



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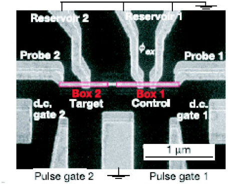
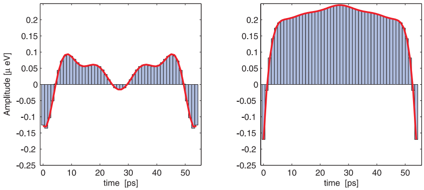
set-up





$\Rightarrow$ timeopt. CNOT: some 5 times faster than NEC group

- Quality $q := F e^{-\tau_{\rm op}/\tau_Q}$

  error $1 - q = 1 - 0.999999999\, e^{-55\text{ps}/10\text{ns}} = 0.0055$

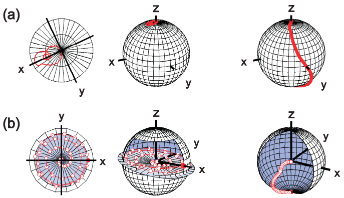
  (NEC: $1 - q = 1 - 0.4188\, e^{-250\text{ps}/10\text{ns}} = 0.5917$)

PRA **75**, 012302 (2007)

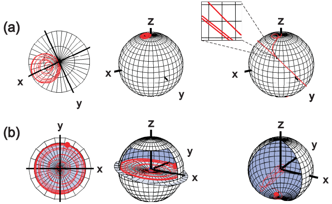
Time-Optimal Quantum Control

Realising Quantum Gates for Charge Qubits

- time-optimal



- NEC pioneer group



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PRA **75**, 012302 (2007)

Goal: realise *timeoptimal* CNOT on 2 coupled charge qubits

- pseudospin Hamiltonian: $H = H_{\text{drift}} + H_{\text{control}}$

$$
\begin{aligned}
H_{\text{drift}} = &-\left(\frac{E_m}{4} + \frac{E_{c1}}{2}\right)(\sigma_z^{(1)} \otimes \mathbf{1}) - \frac{E_{J1}}{2}(\sigma_x^{(1)} \otimes \mathbf{1}) \\
&-\left(\frac{E_m}{4} + \frac{E_{c2}}{2}\right)(\mathbf{1} \otimes \sigma_z^{(2)}) - \frac{E_{J2}}{2}(\mathbf{1} \otimes \sigma_x^{(2)}) \\
&+\frac{E_m}{4}(\sigma_z^{(1)} \otimes \sigma_z^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
H_{\text{control}} = &\left(\frac{E_m}{2} n_{g2} + E_{c1} n_{g1}\right)(\sigma_z^{(1)} \otimes \mathbf{1}) \\
&+\left(\frac{E_m}{2} n_{g1} + E_{c2} n_{g2}\right)(\mathbf{1} \otimes \sigma_z^{(2)})
\end{aligned}
$$

NB: components $\{H_d + H_d, H_c\}$ form minimal generating set of $\mathfrak{su}(4)$.

- Symmetry: real symmetric Hamiltonians
  - ⇒ palindromic controls for self-inverse gates (CNOT)
  - ⇒ composed of cos Fourier series
  - ⇒ Cauer synthesis by LC elements (no resistive R)

PRA **75**, 012302 (2007)

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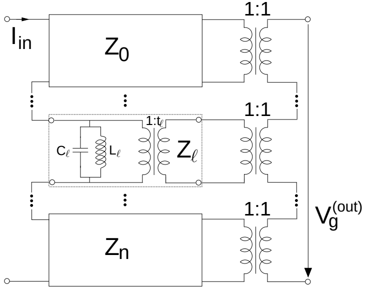
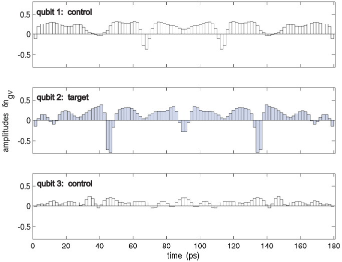


PRA **75**, 012302 (2007)

Goal: TOFFOLI gate on 3 linearly coupled charge qubits



13 times faster than NEC

- error rates cut by two orders of magnitude ($T_2 \simeq 10$ ns):

1. direct gate by optimal control
$$1 - q = 1 - 0.99999\, e^{-180\text{ps}/10\text{ns}} = 0.0178$$
2. by 9 CNOT's from optimal control
$$1 - q = 1 - (0.999999999\, e^{-55\text{ps}/10\text{ns}})^9 = 0.0483$$
3. by 9 CNOT's under pioneering controls
$$1 - q = 1 - (0.4188\, e^{-250\text{ps}/10\text{ns}})^9 = 0.9997$$

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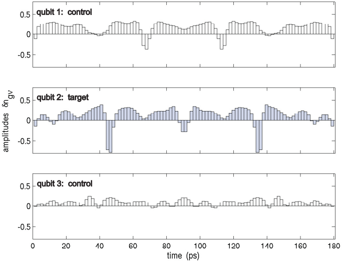

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Examples of Quantum Control

Realising Quantum Gates for Charge Qubits

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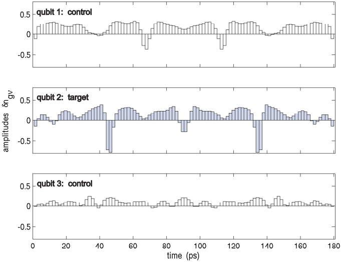

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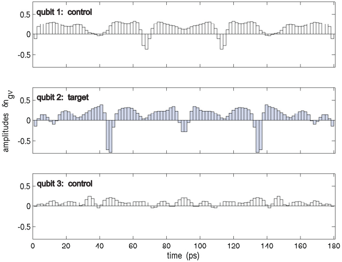

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Principles: Optimal Quantum Control

Scope in Optimal Control:
maximise quality function **subject to** equation of motion

Scenarios:

- Hamiltonian dynamics

  notation: $U := e^{-itH}$; $\mathrm{Ad}_U(\cdot) := U(\cdot)U^{-1}$; $\mathrm{ad}_H(\cdot) := [H, \cdot\,]$

  1. pure state $\quad |\dot{\psi}\rangle = -iH\,|\psi\rangle \quad \in \mathcal{H}$
  2. gate $\quad \dot{U} = -iH\,U \quad \in \mathcal{U}(\mathcal{H})$
  3. non-pure state $\quad \dot{\rho} = -i\,\mathrm{ad}_H\,(\rho) \quad \in \mathcal{B}_1(\mathcal{H})$
  4. projective gate $\quad \dot{\mathrm{Ad}}_U = -i\,\mathrm{ad}_H \circ \mathrm{Ad}_U \quad \in \mathcal{U}(\mathcal{B}_1(\mathcal{H}))$

- Master equations of dissipative dynamics

  3'. non-pure state $\quad \dot{\rho} = -(i\,\mathrm{ad}_H + \Gamma)\,(\rho)$

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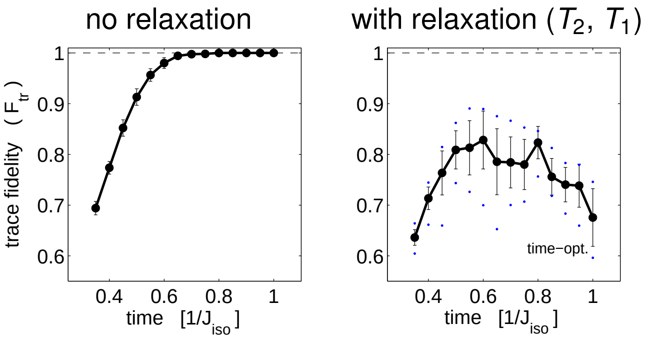
- **System-II**: driving outside slowly-relaxing subspace



- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently

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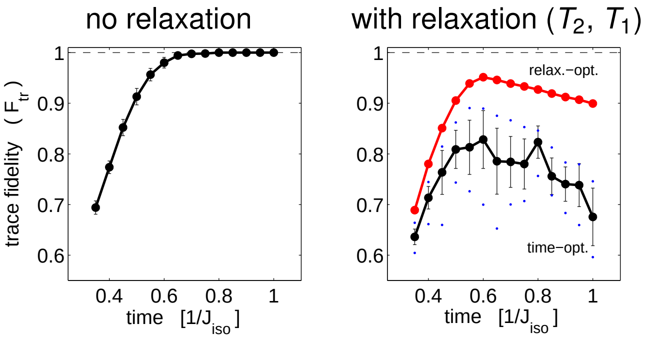
- **System-II**: driving outside slowly-relaxing subspace

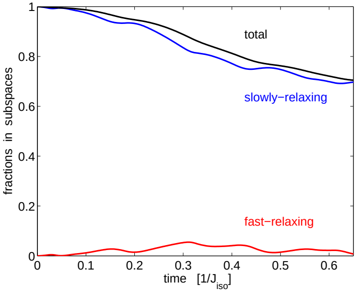


- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently
- relaxation-optimised: systematic substantial gain

JPB **44** 154013 (2011), quant-ph/0609037

CNOT under **System-II**: Projection into Subspaces

- time-optimised



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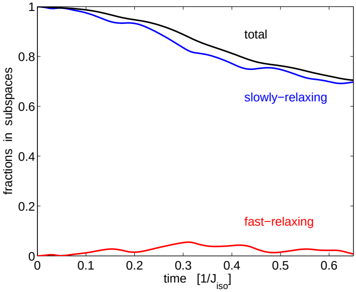
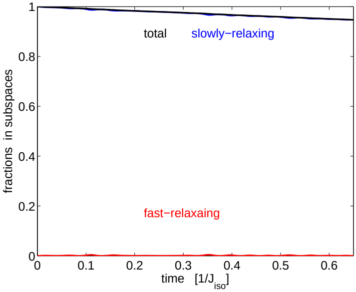
CNOT under **System-II**: Projection into Subspaces

- time-optimal
- opt. against decoherence





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CNOT under **System-II**: Projection into Subspaces

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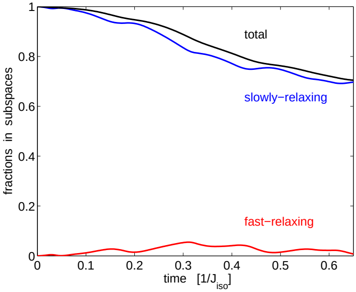
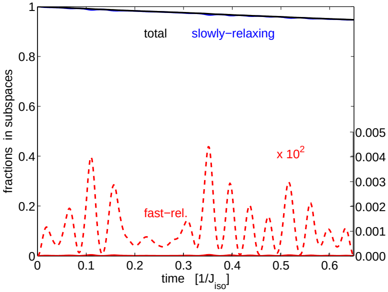
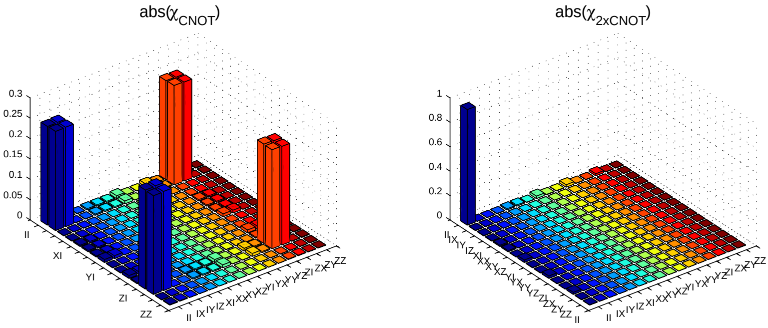




- CNOT under **System-II**: Process Tomography of Gate Protected against Dissipation by Optimal Control

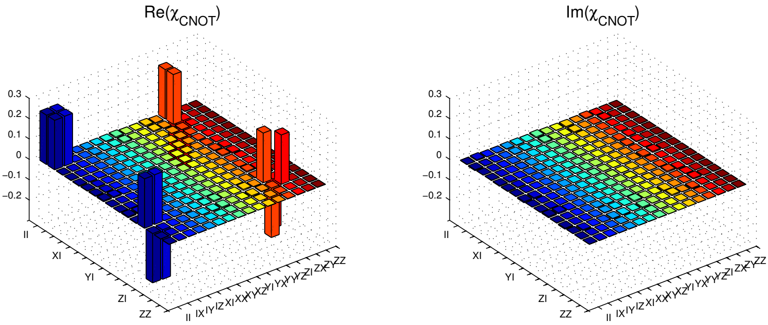


Examples of Quantum Control

3. Realising Quantum Gates with Minimal Relaxation

JPB **44** 154013 (2011), quant-ph/0609037

- CNOT under **System-II**: Process Tomography of Gate Protected against Dissipation by Optimal Control

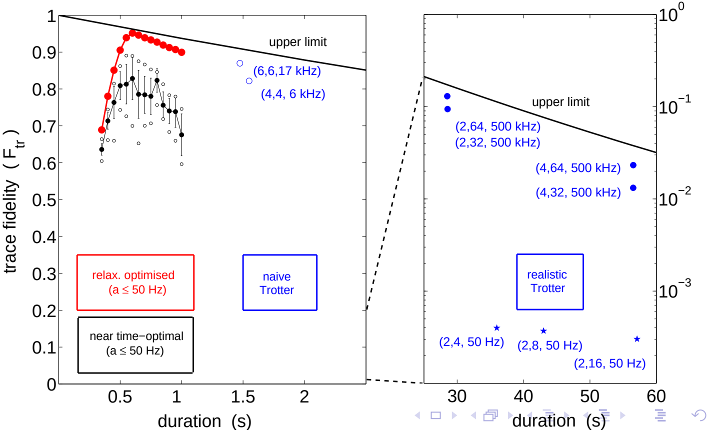


JPB **44** 154013 (2011), quant-ph/0609037

CNOT under **System-II**: comparison of methods

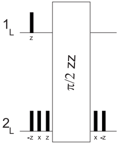
by decoherence control: $>$ 95% fideltity

conventional: $<$ 15% fidelity



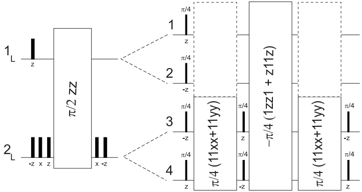
Decoherence-Protected CNOT-Gate via

- logical qubits



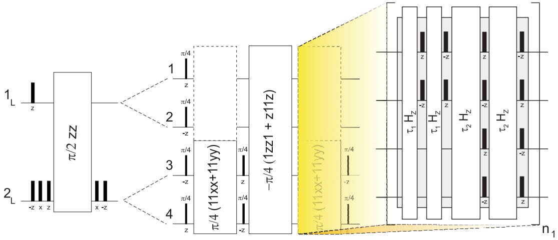
Decoherence-Protected CNOT-Gate via

■ physical qubits



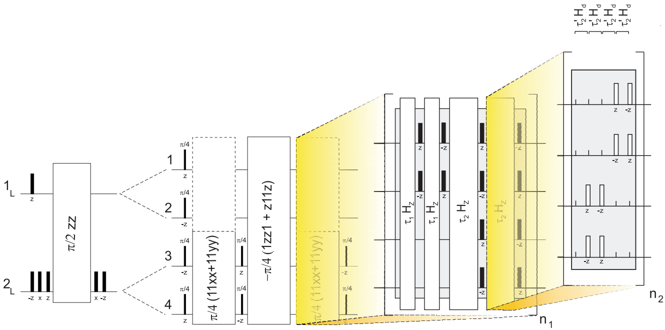
Decoherence-Protected CNOT-Gate via

- realisation by **System-I**



Decoherence-Protected CNOT-Gate via

- realisation by **System-II**



Model:
qubit coupled to a two-level fluctuator coupled to a bath

$$H = H_S + H_I + H_B$$

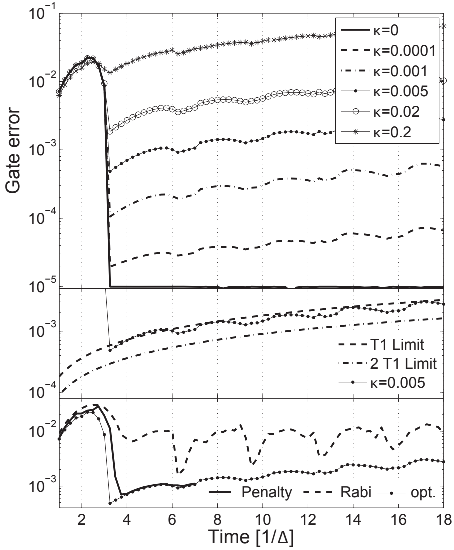
- $H_S = E_1(t)\sigma_z + \Delta\sigma_x + E_2\tau_z + \Lambda\sigma_z\tau_z$
- $H_I = \sum_i \lambda_i(\tau^+ b_i + \tau^- b_i^\dagger)$
- $H_B = \sum_i \hbar\omega_i b_i^\dagger b_i$

Ohmic bath spectrum: $J(\omega) = \sum_i \lambda_i^2 \delta(\omega - \omega_i) = \kappa\omega\Theta(\omega - \omega_c)$

couplings λ_i, damping κ, high-freq. cut-off ω_c

PRL **102** 090401 (2009)

with P. Rebentrost and F. Wilhelm



←RABI pulse
←cut error by factor ≤ 10 with optimal control

PRL **102** 090401 (2009)

- Principle: embed to Markovian and project

$$
\begin{array}{ccc}
\rho_0 = \rho_{SE}(0) \otimes \rho_B(0) & \xrightarrow{\quad \mathrm{Ad}_{W}(t) \quad} & \rho(t) = W(t)\rho_0 W^\dagger(t) \\
\Pi_{SE} \Big\downarrow \mathrm{tr}_B & & \Pi_{SE} \Big\downarrow \mathrm{tr}_B \\
\rho_{SE}(0) & \xrightarrow[\text{Markovian}]{\quad F_{SE}(t) \quad} & \rho_{SE}(t) \\
\Pi_S \Big\downarrow \mathrm{tr}_E & & \Pi_S \Big\downarrow \mathrm{tr}_E \\
\rho_S(0) & \xrightarrow[non-Markovian]{\quad F_S(t) \quad} & \rho_S(t)
\end{array}
$$

Lindbladians Generate (Lie) Semigroup

Rep. Math. Phys. **64** 93 (2009)

Consider: controlled system with *time dep* Lindbladians $\{\mathcal{L}_u(t)\}$

$$\dot{X} = -\mathcal{L}_u(t)X = -(iH_d + i\sum_j u_j(t)H_j + \Gamma)X$$

Lindbladians $\{\mathcal{L}_u\}$ form

- *Lie wedge* $\mathfrak{w}$
- *Lie semialgebra* $\mathfrak{w}_{\mathfrak{s}}$, if $\{\mathcal{L}_u\}$ BCH compatible with $\mathfrak{w}$

  i.e. $L_j * L_k := L_j + L_k + \frac{1}{2}[L_j, L_k] + \dots \in \mathfrak{w}$

  then $\{e^{-t\mathcal{L}_{\text{eff}}} \mid t > 0\}$ physical at all times.
- Else $\{e^{-t\mathcal{L}_{\text{eff}}} \mid t > 0\}$ unphysical except $t = 0$; $t = t_{\text{eff}}$ etc.

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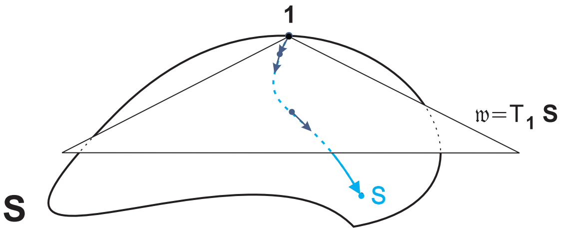

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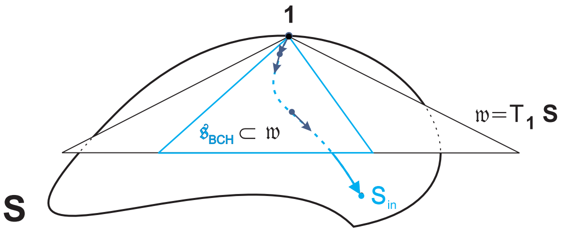

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Corollary (*cave:* ‘woodcut’, details in Rep. Math. Phys. **64** (2009) 93.)

- *A channel is (time dependent) Markovian, iff there is representation $T = e^{-\mathcal{L}_1} e^{-\mathcal{L}_2} \cdots e^{-\mathcal{L}_r}$ so that the $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$ generate a Lie wedge $\mathfrak{w}_r$.*
- *Moreover, T specialises to time independent form, iff its Lie wedge $\mathfrak{w}_r$ specialises to a Lie semialgebra.*

Complements recent work: Wolf,Cirac, *Commun. Math. Phys.* (2008) & Wolf,Eisert,Cubitt,Cirac, *PRL* (2008)

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Consider bilinear control system

$$\dot{X} = -(A + \sum_j u_j B_j)X \text{ with } A := i\widehat{H}_d + \Gamma_L \text{ and } B_j := i\widehat{H}_j$$

- controllability condition for closed systems: $\langle iH_d, iH_j \,|\, j = 1, 2, \dots m\rangle_{\text{Lie}} = \mathfrak{su}(N)$
- WH-condition for open systems: $\langle iH_d, iH_j \,|\, j = 1, 2, \dots m\rangle_{\text{Lie}} = \mathfrak{su}(N)$
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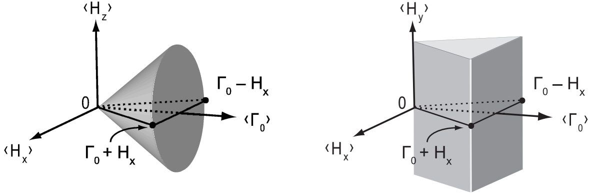

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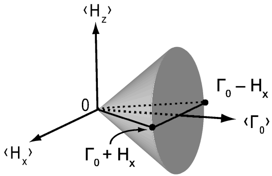

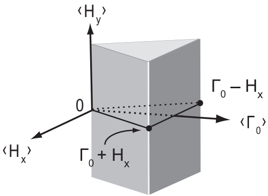

I. Q-Control
II. Q-Simulation
III. Algorithms
IV. Applications
Outlook
Conclusions
A1
A2
A3

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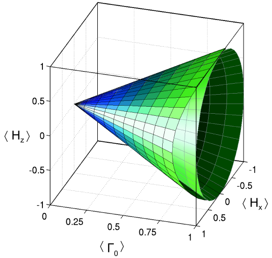

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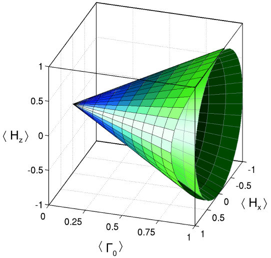

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Conclusions

Exploit *symmetries* for the Royal Road to:

1. Controllability
   - absence of symmetry plus inclusion fermionic & bosonic systems
2. Simulability
   - efficient q-simulation: algebraic understanding
3. DYNAMO Modular Platform
4. Lie Semigroups
   - as new *unifying framework* for open systems

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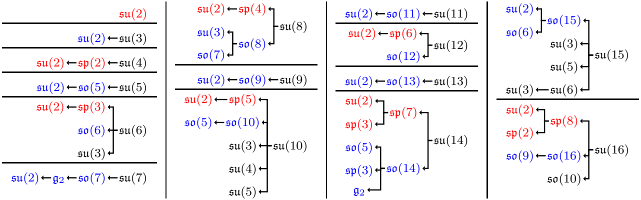

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Acknowledgements

Thanks go to:

Robert Zeier
Shai Machnes, Uwe Sander, Pierre de Fouquières, Sophie Schirmer
Gunther Dirr, Corey O'Meara

integrated EU programme; excellence network; high-speed parallel cluster

References:

J. Magn. Reson. **172**, 296 (2005), *PRA* **72**, 043221 (2005), *PRA* **84**, 022305 (2011)

PRA **75**, 012302 (2007); *PRL* **102** 090401 (2009), *JPB* **44**, 154013 (2011)
Rev. Math. Phys. **22**, 597 (2010), *Rep. Math. Phys.* **64**, 93 (2009);
PRA **81**, 032319 (2010); *PRB* **81**, 085328 (2010);

arXiv:0904.4654, IEEE Proc. ISCCSP 2010 23.2, Proc. MTNS, 2341 (2010),

arXiv: 1012.5256, arXiv: 1103.2703

Design Rules

For an *n* spin-$\frac{1}{2}$ system with a connected coupling graph and no symmetries to be fully controllable it suffices that

(1) the coupling is Ising-*ZZ* and *each qubit* belongs to a *type* that is jointly operator controllable locally,

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Example (inner symmetries)

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(a)	XX_{12}	0	0	10	$\sum_i \sigma_z^{(i)}$	$\mathfrak{so}(5)$
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(c)	Z_1	1	0	25	$\sum_i \sigma_z^{(i)}$	$\mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1))$
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Minimalistically Controlled Systems

Examples

quant-ph/0904.4654

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(a)	XX_{12}	0	0	10	$\sum_i \sigma_z^{(i)}$	$\mathfrak{so}(5)$
(b)	XY_{12}	0	0.3	20	$\prod_i \sigma_z^{(i)}$	$\mathfrak{so}(5) \hat{\oplus} \mathfrak{so}(5)$
(c)	Z_1	1	0	25	$\sum_i \sigma_z^{(i)}$	$\mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1))$
(d)	Z_1	1	0.3	45	$\prod_i \sigma_z^{(i)}$	$\mathfrak{so}(10)$
(e)	XX_{12}	1	0	11	$\sum_i \sigma_z^{(i)}$	$\mathfrak{s}(\mathfrak{o}(5) \oplus \mathfrak{u}(1))$
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(f')	Z_1, X_1, XY_{12}	1	0.3	55	$\lambda \cdot \mathbb{1}$	$\mathfrak{so}(11)$
(g)	Z_1, X_1, XXX_{12}	0	0	1023	$\lambda \cdot \mathbb{1}$	$\mathfrak{su}(2^5)$
(h)	Z_1, X_1, XYZ_{12}	1	0.3	1023	$\lambda \cdot \mathbb{1}$	$\mathfrak{su}(2^5)$

$$H_d := \tfrac{1}{2}\sum_{k=2}^{n-1}(1+\gamma)X_kX_{k+1} + (1-\gamma)Y_kY_{k+1} + \sum_{i=2}^{n} B_i\, Z_i$$

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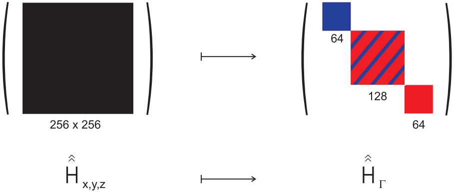
Control of Markovian Open Systems

beyond Decoherence-Free Subspaces (DFS)

Original Principle:
Code *logical* qubits in decoherence-free *physical* levels

- master equation: $\dot{\rho} = -(i\,\mathrm{ad}_H + \Gamma)\,\rho$
- DFS: eigenspace to Γ with eigenval =0 (Bell states $\mathcal{B}$)
- Express $\widehat{H} \equiv \mathrm{ad}_H$ in eigenbasis of Γ (here 4 qubits)



- Task: perform calculation (e.g. CNOT) within DFS

Zanardi, Rasetti, *PRL* **79** (1997), 3309.
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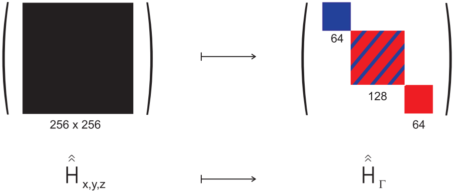

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- 1 logical qubit coded by 2 physical qubits in Bell states

$$|0\rangle_L := |\psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\ ,\ |1\rangle_L := |\psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

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Control of Markovian Open Systems

System of 2 Qubits Coded in 4 Spins

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- 2 logical qubits coded by 4 physical qubits

- protection against T_2 relaxation (Redfield: $\Gamma \sim [ZZ, [ZZ, \rho]]$)

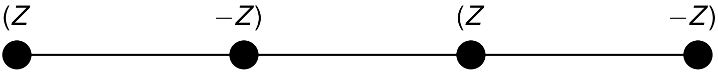
because $[\rho, ZZ] = 0 \quad \forall \quad \rho \in \mathcal{B} \otimes \mathcal{B}$

- controls

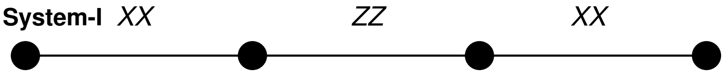
Control of Markovian Open Systems

Model with 4 Linearly Coupled Spins

- controls



- drift: Ising (ZZ) and Heisenberg (XX) interactions



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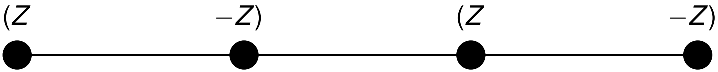

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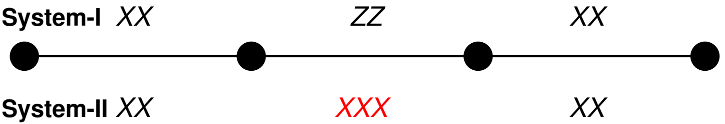

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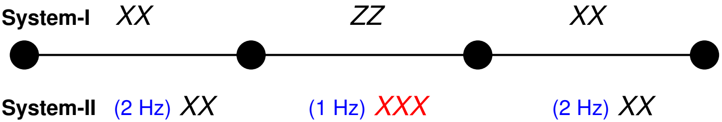

- relaxation ($T_2^{-1} : T_1^{-1} = 4.0\ \mathrm{s}^{-1} : 0.024\ \mathrm{s}^{-1} \simeq 170 : 1$)

- **System-I**: staying within slowly-relaxing subspace
  - drift Hamiltonian D_1 with Ising-ZZ
  - controls $C_{1,2}$

$$
\begin{aligned}
D_1 &:= J_{xx}\,(xx\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}xx + yy\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}yy) + J_{zz}\,\mathbb{1}zz\mathbb{1}\\
C_1 &:= z\mathbb{1}\mathbb{1}\mathbb{1} - \mathbb{1}z\mathbb{1}\mathbb{1}\\
C_2 &:= \mathbb{1}\mathbb{1}z\mathbb{1} - \mathbb{1}\mathbb{1}\mathbb{1}z\,.\\
&\Rightarrow \langle D_1, C_1, C_2\rangle_{\text{Lie}}\big|_{\mathcal{B}\otimes\mathcal{B}} \overset{\text{rep}}{=} \mathfrak{su}(4)
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- **System-II**: driving outside slowly-relaxing subspace
  - drift: extended to isotropic Heisenberg-XXX

$$D_1 + D_2 \;:=\; J_{xx}\left(xx\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}xx + yy\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}yy\right) + J_{xyz}\left(\mathbb{1}xx\mathbb{1} + \mathbb{1}yy\mathbb{1} + \mathbb{1}zz\mathbb{1}\right)$$

  - Lie-algebraic closure: in 66-dim. Lie algebra

$$\dim\langle (D_1 + D_2), C_1, C_2\rangle_{\mathrm{Lie}} = 66 \quad [\stackrel{\mathrm{iso}}{=} \mathfrak{so}(12)]$$

  - $\mathfrak{su}(4)$ merely subalgebra

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I. Q-Control
II. Q-Simulation
III. Algorithms
IV. Applications
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Conclusions
A1
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Systems I, II
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A3

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System-II:

- full controllability within slowly-relaxing subspace
  - observation

$$e^{-i\pi C_1}(D_1 + D_2)e^{i\pi C_1} = D_1 - D_2$$

  - Trotter limit

$$\lim_{n\to\infty}\left(e^{-i(D_1+D_2)/(2n)}e^{-i(D_1-D_2)/(2n)}\right)^n = e^{-iD_1}$$

  - reduction of dynamics

$$\text{System-II} \xrightarrow{\text{infinite \# switchings}} \text{System-I}$$

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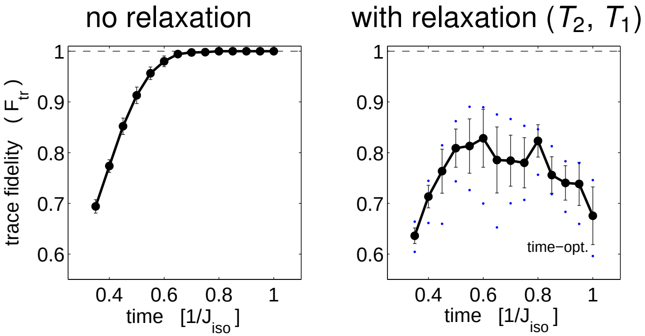
Typical: system drives outside protected subspace



- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently

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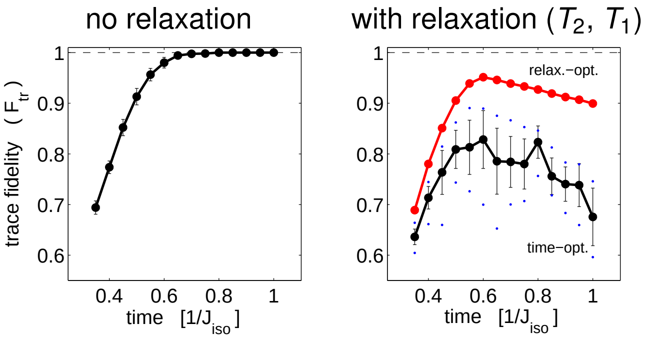
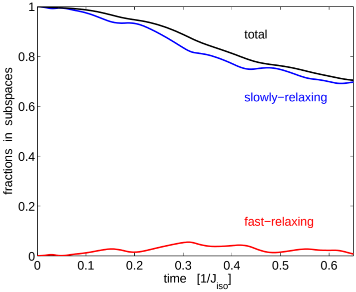


- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently
- relaxation-optimised: systematic substantial gain

quant-ph/0609037

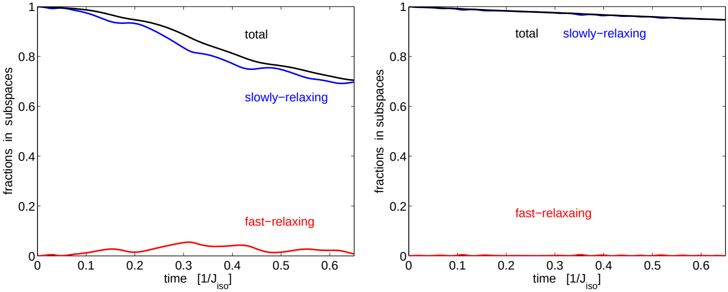
CNOT: Projection into Subspaces

- time-optimised



CNOT: Projection into Subspaces

- time-optimsed
- opt. against decoherence



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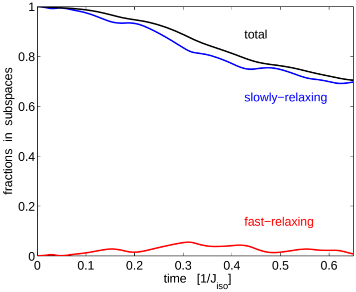
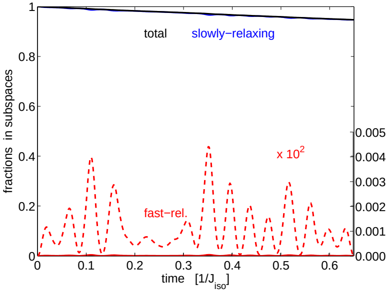
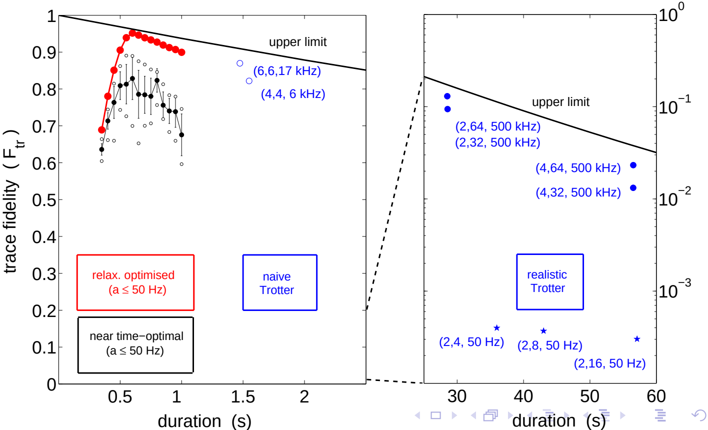




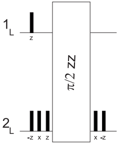
CNOT: comparison of methods

mbox by decoherence control: $>$ 95% fideltity conventional: $<$ 15% fidelity



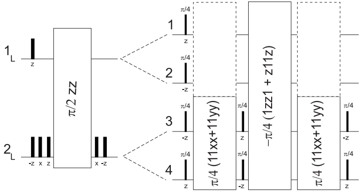
Decoherence-Protected CNOT-Gate via

- logical qubits



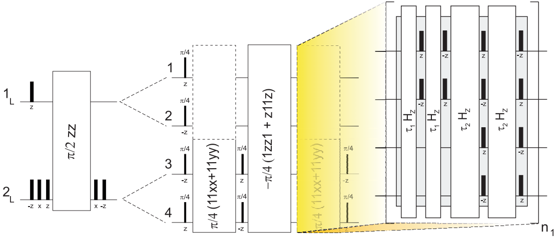
Decoherence-Protected CNOT-Gate via

- physical qubits



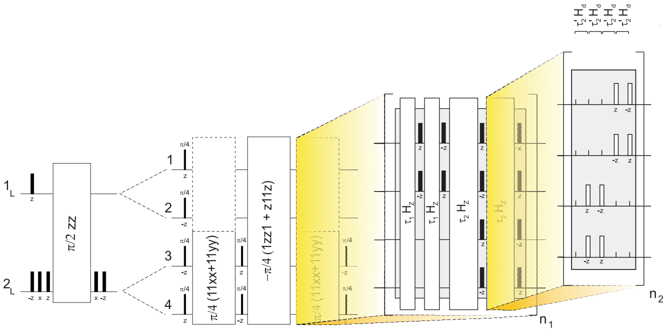
Decoherence-Protected CNOT-Gate via

- realisation by **System-I**



Decoherence-Protected CNOT-Gate via

- realisation by **System-II**



Model:
qubit coupled to a two-level fluctuator coupled to a bath

$$H = H_S + H_I + H_B$$

- $H_S = E_1(t)\sigma_z + \Delta\sigma_x + E_2\tau_z + \Lambda\sigma_z\tau_z$
- $H_I = \sum_i \lambda_i(\tau^+ b_i + \tau^- b_i^\dagger)$
- $H_B = \sum_i \hbar\omega_i b_i^\dagger b_i$

  Ohmic bath spectrum: $J(\omega) = \sum_i \lambda_i^2 \delta(\omega - \omega_i) = \kappa\omega\Theta(\omega - \omega_c)$

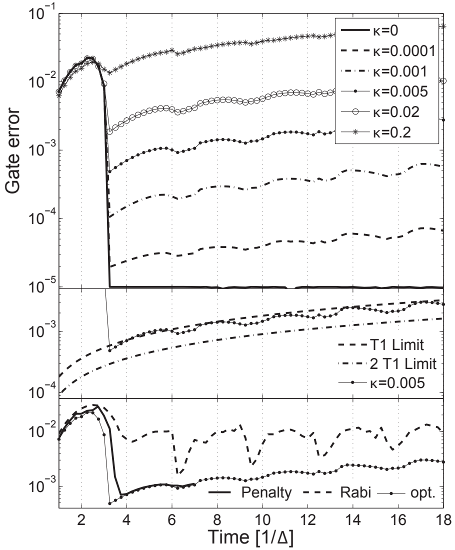
  couplings λ_i, damping κ, high-freq. cut-off ω_c

PRL **102** 090401 (2009)

Control of Non-Markovian Open Systems

Qubit Coupled via Two-Level Fluctuator to Spin Bath

with P. Rebentrost and F. Wilhelm



←RABI pulse

←cut error by factor $\leq$ 10 with optimal control

PRL **102** 090401 (2009)

Control of Non-Markovian Open Systems

- Principle: embed to Markovian and project

$$
\begin{array}{ccc}
\rho_0 = \rho_{SE}(0) \otimes \rho_B(0) & \xrightarrow{\mathrm{Ad}_{W}(t)} & \rho(t) = W(t)\rho_0 W^\dagger(t) \\
\Pi_{SE} \big\downarrow \mathrm{tr}_B & & \Pi_{SE} \big\downarrow \mathrm{tr}_B \\
\rho_{SE}(0) & \xrightarrow[\text{Markovian}]{F_{SE}(t)} & \rho_{SE}(t) \\
\Pi_{S} \big\downarrow \mathrm{tr}_E & & \Pi_{S} \big\downarrow \mathrm{tr}_E \\
\rho_{S}(0) & \xrightarrow[\text{non}-\text{Markovian}]{F_{S}(t)} & \rho_{S}(t)
\end{array}
$$

- Viewing Markovian Quantum Channels as Lie Semigroups with GKS-Lindblad Generators as Lie Wedge

Observe: two notions

Definition

- A CP-Map T is *(infinitely) divisible*, if $\forall r \in \mathbb{N}$ there is a S with $T = S^r$.
- A CP-map T is *infinitesimally divisible* if $\forall \epsilon > 0$ there is a sequence $\prod_{j=1}^{r} S_j = T$ with $||S_j - \mathrm{id}|| \leq \epsilon$.

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Notions:
time-(in)dependent CP-map: solution of
time-(in)dependent master eqn. $\dot{X} = -\mathcal{L} \circ X$.

Theorem (Wolf & Cirac (2008))

- *The set of all time-independent Markovian CP-maps coincides with the set of all (infinitely) divisible CP-maps.*
- *The set of all time-dependent Markovian CP-maps coincides with the closure of the set of all infinitesimally divisible CP-maps.*

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Observe: semigroup structure

$$\mathrm{Reach}(\mathbb{1}, t_1) \circ \mathrm{Reach}(\mathbb{1}, t_2) = \mathrm{Reach}(\mathbb{1}, t_1 + t_2) \; \forall t_\nu \geq 0$$

Definition

- A *subsemigroup* $\mathbf{S} \subset \mathbf{G}$ of a Lie group $\mathbf{G}$ with algebra $\mathfrak{g}$ contains $\mathbb{1}$ and follows $\mathbf{S} \circ \mathbf{S} \subseteq \mathbf{S}$. Its largest subgroup is denoted $E(\mathbf{S}) := \mathbf{S} \cap \mathbf{S}^{-1}$.
- Its *tangent cone* is defined by

$$\mathrm{L}(\mathbf{S}) := \{\dot{\gamma}(0) \mid \gamma(0) = \mathbb{1}, \gamma(t) \in \mathbf{S}, t \geq 0\} \subset \mathfrak{g},$$

for any $\gamma : [0, \infty) \to \mathbf{G}$ being a smooth curve in $\mathbf{S}$.

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- A *wedge* $\mathfrak{w}$ is a closed convex cone of a finite-dim. real vector space.
- Its *edge* $E(\mathfrak{w}) := \mathfrak{w} \cap \text{-}\mathfrak{w}$ is the largest subspace in $\mathfrak{w}$.
- It is a *Lie wedge* if it is invariant under conjugation

$$e^{\mathrm{ad}_g}(\mathfrak{w}) \equiv e^{g} \mathfrak{w} e^{-g} = \mathfrak{w}$$

  for all edge elements $g \in E(\mathfrak{w})$.
- A *Lie semialgebra* is a Lie wedge compatible with BCH multiplication $X * Y := X + Y + \frac{1}{2}[X, Y] + \dots$ so that for a BCH neighbourhood B of $0 \in \mathfrak{g}$

$$(\mathfrak{w} \cap B) * (\mathfrak{w} \cap B) \in \mathfrak{w} \,.$$

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- A *Lie semialgebra* is a Lie wedge compatible with BCH multiplication $X * Y := X + Y + \frac{1}{2}[X, Y] + \dots$ so that for a BCH neighbourhood B of $0 \in \mathfrak{g}$
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- A *wedge* $\mathfrak{w}$ is a closed convex cone of a finite-dim. real vector space.
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Define as completely positive, trace-preserving invertible linear operators the set $\mathbf{P}^{\mathbf{cp}}$, and let $\mathbf{P}_0^{cp}$ denote the connected component of the unity.

Theorem (Kossakowski, Lindblad)

The Lie wedge to the connected component of the unity of the semigroup of all invertible CPTP *maps is given by the set of all linear operators of GKS-Lindblad form:*

$$L(\mathbf{P}_0^{cp}) = \{-\mathcal{L} | \mathcal{L} = -(i\,\mathrm{ad}_H + \Gamma_L)\} \quad \textit{with}$$

$$\Gamma_L(\rho) = \tfrac{1}{2}\sum_k \{V_k^\dagger V_k, \rho\}_+ - 2V_k \rho V_k^\dagger$$

GKS-Lindblad Operators as Lie Wedge

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Theorem

The semigroup

$$\mathbf{T} := \overline{\left\langle \exp\left(\mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})\right)\right\rangle_S} \subseteq \mathbf{P}_0^{\mathrm{cp}}$$

generated by $\mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})$ *is a Lie subsemigroup with global Lie wedge* $\mathrm{L}(\mathbf{T}) = \mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})$.

Corollary (to Wolf, Cirac (2008))

$\mathbf{P}_0^{\rm cp}$ *itself is not a Lie subsemigroup, yet it comprises*

(1) *the set of time independent Markovian channels, i.e. the union of all one-parameter Lie semigroups* $\{\exp(-\mathcal{L}t) \mid t \geq 0\}$ *with* $\mathcal{L}$ *in GKS-Lindblad form;*

(2) *the closure of the set of time dependent Markovian channels, i.e. the Lie semigroup* $\mathbf{T}$*;*

(3) *a set of non-Markovian channels whose intersection with* $\mathbf{P}_0^{\rm cp}$ *has non-empty interior.*

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Corollary

A quantum channel is time dependent Markovian iff it allows for a representation $T = \prod_{j=1}^{r} S_j$, *where* $S_1 = e^{-\mathcal{L}_1}, S_2 = e^{-\mathcal{L}_2}, \ldots, S_r = e^{-\mathcal{L}_r}$ *so that there is a global Lie wedge* $\mathfrak{w}_r$ *generated by* $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$.

Corollary

Let $T = \prod_{j=1}^{r} S_j$ be a time dependent Markovian channel with $S_1 = e^{-\mathcal{L}_1}, S_2 = e^{-\mathcal{L}_2}, \dots, S_r = e^{-\mathcal{L}_r}$ and let $\mathfrak{w}_r$ denote the smallest global Lie wedge generated by $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$. Then

- *T boils down to a time independent Markovian channel, if it is sufficiently close to the unity and if there is a representation so that the associated Lie wedge $\mathfrak{w}_r$ specialises to a Lie semialgebra.*

Complements recent work: Wolf,Cirac, *Commun. Math. Phys.* (2008) & Wolf,Eisert,Cubitt,Cirac, *PRL* (2008)

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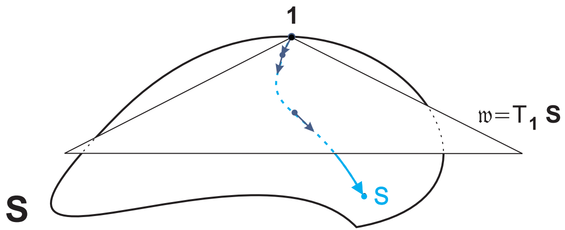
Consider: controlled system with *time dep* Liouvillians $\{\mathcal{L}_u(t)\}$

$$\dot{X} = -\mathcal{L}_u(t)X = -(iH_d + i\sum_j u_j(t)H_j + \Gamma)X$$



Liouvillians $\mathcal{L}_u$ form

- *Lie wedge* $\mathfrak{w}$
- *Lie semialgebra* $\mathfrak{s} \subset \mathfrak{w}$ if $\{\mathcal{L}_u\}$ BCH compatible with $\mathfrak{w}$
  then $\{e^{-t\mathcal{L}_{\text{eff}}} \mid t > 0\}$ physical at all times.
- Else $\{e^{-t\mathcal{L}_{\text{eff}}} \mid t > 0\}$ unphysical except $t = 0$; $t = t_{\text{eff}}$ etc.

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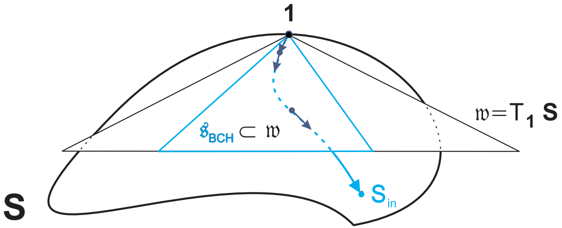

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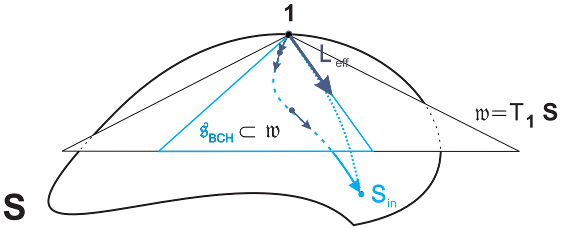

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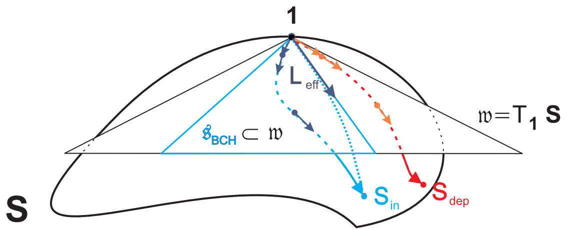

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