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Precision bounds for gradient magnetometry with atomic ensembles

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March 5, 2018

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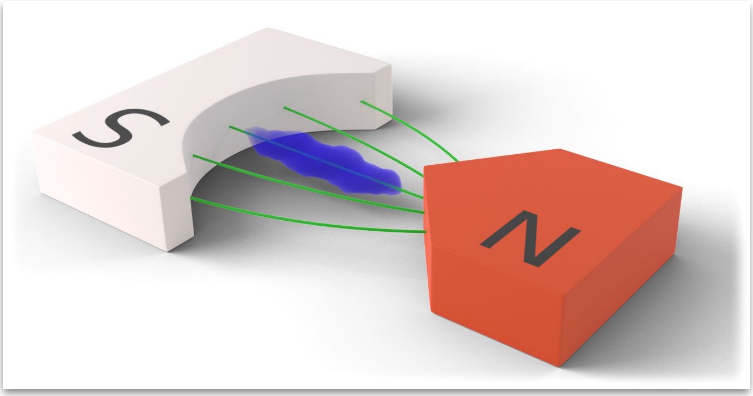
Introduction

- We study achievable precisions of the estimation of the **MAGNETIC FIELD GRADIENT**.

- We are interested in the **SCALING** with the particle number, *N*.

- We study **PRECISION BOUNDS** for:

1. Spin-chains
2. Double well experiments
3. Single cloud of atoms



*Related work: S. Altenburg et al. Phys. Rev. A **96**, 042319 (2017)*

$$\boldsymbol{B}(x,0,0) = \boldsymbol{B}_0 + x\boldsymbol{B}_1 + \mathcal{O}(x^2)$$

Homogeneous part (→ $\boldsymbol{B}_0$)

Gradient parameter (→ $\boldsymbol{B}_1$)

The *N*-particle state and the setup

Factorizable between the spatial and spin parts:

$$\varrho = \varrho^{(\mathrm{x})} \otimes \varrho^{(\mathrm{s})}$$

Spatial part:

$$\varrho^{(\mathrm{x})} = \int \frac{P(\boldsymbol{x})}{\langle \boldsymbol{x} | \boldsymbol{x} \rangle} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \, \mathrm{d}\boldsymbol{x}$$

Spin chains, cold atomic ensembles, double well experiments, …

- Each particle interacts with the magnetic field:

$$h^{(n)} = \gamma B_z^{(n)} \otimes j_z^{(n)}$$

- The gradient parameter B_1 is encoded in the phase b_1:

$$U = e^{-i(b_0 H_0 + b_1 H_1)}$$

- Generators of phase-shifts:

$$H_0 = \sum_{n=1}^{N} j_z^{(n)} = J_z$$

$$H_1 = \sum_{n=1}^{N} x^{(n)} j_z^{(n)}$$

Cramér-Rao Precision Bounds

- Fos states <u>INSENSITIVE</u> to the homogeneous fields:

$$(\Delta b_1)^{-2}|_{\max} = \mathcal{F}_\mathrm{Q}[\varrho, H_1, H_1] = \mathcal{F}_\mathrm{Q}[\varrho, H_1]$$

- Fos states <u>SENSITIVE</u> to the homogeneous fields: $\mathcal{F}_{ij} := \mathcal{F}_\mathrm{Q}[\varrho, H_i, H_j]$

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_{11} - \frac{\mathcal{F}_{01}\mathcal{F}_{10}}{\mathcal{F}_{00}}$$

Quantum Fisher Information (QFI)

$$\mathcal{F}_\mathrm{Q}[\varrho, A, B] := 2\sum_{k,k'} \frac{(p_k - p_{k'})^2}{p_k + p_{k'}} A_{k,k'} B_{k',k}$$

States insensitive to homogeneous fields (H_0)

The matrix elements of H_1

$$(H_1)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu} = \delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N} x_n j^{(n)}|\nu\rangle$$

Precision bound for the **gradient** parameter

$$(\Delta b_1)^{-2}|_{\max} = \sum_{n,m}^{N}\int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, j_z^{(m)}]$$

Note: *This bound is **invariant** under spatial translations.*

States sensitive to homogeneous fields (*H*$_0$)

The matrix elements of H_1

$$(H_0)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu} = \delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N} j_z^{(n)}|\nu\rangle$$

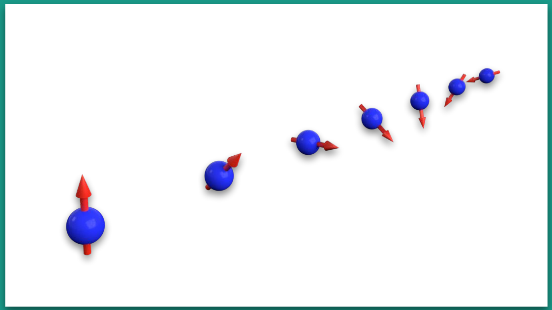
Precision bound for the **gradient** parameter

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m}^{N} \int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, j_z^{(m)}] - \frac{\left(\sum_{n=1}^{N}\int x_n P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}, J_z]\right)^2}{\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, J_z]}$$

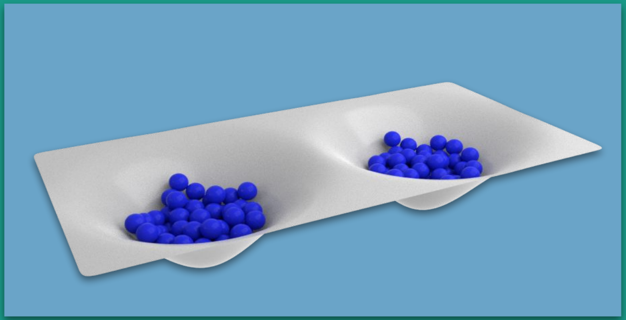
Note: *This bound is also **invariant** under spatial translations.*

Spin Chain and Double Well

$$P(\boldsymbol{x}) = \prod_{n=1}^{N} \delta(x_n - na)$$



$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n + a) \prod_{n=N/2+1}^{N} \delta(x_n - a)$$



Gradient magnetometry with spin chains

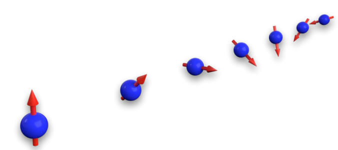
State totally polarized in the y direction: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$

PRECISION BOUND:

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_{11} - \frac{\mathcal{F}_{01}\mathcal{F}_{10}}{\mathcal{F}_{00}} \qquad \sigma_{\rm ch}^2 = a^2 \frac{N^2-1}{12}$$

$$(\Delta b_1)^{-2}|_{\max} = 2\sigma_{\rm ch}^2 N j$$

<u>Note</u>: *Even if the state is sensitive to the homogeneous field,* ***all bounds*** *we show in this work are* ***SATURABLE****. See* <u>*arXiv.org:1703.09056*</u>



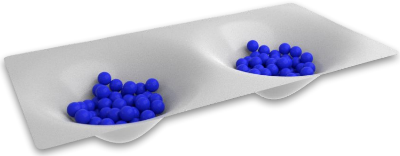
Double wells for gradient estimation

The variance for double wells: $\sigma_{\mathrm{dw}}^2 = a^2$

The **entangled** state that saturates the *Heisenberg limit*:

$$|\psi\rangle = \frac{|j \cdots j\rangle^{(\mathrm{L})}|-j \cdots -j\rangle^{(\mathrm{R})} + |-j \cdots -j\rangle^{(\mathrm{L})}|j \cdots j\rangle^{(\mathrm{R})}}{\sqrt{2}}$$

$$(\Delta b_1)^{-2}|_{\max} = 4\sigma_{\mathrm{dw}}^2 N^2 j^2$$



Double wells for gradient estimation

The variance for double wells: $\sigma_{\mathrm{dw}}^2 = a^2$

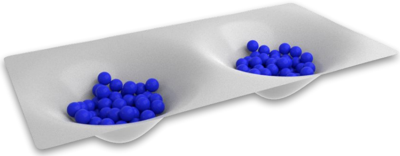
The **entangled** state that saturates the *Heisenberg limit*:

$$|\psi\rangle = \frac{|j\cdots j\rangle^{(\mathrm{L})}|-j\cdots -j\rangle^{(\mathrm{R})} + |-j\cdots -j\rangle^{(\mathrm{L})}|j\cdots j\rangle^{(\mathrm{R})}}{\sqrt{2}}$$

$$(\Delta b_1)^{-2}|_{\max} = 4\sigma_{\mathrm{dw}}^2 N^2 j^2$$

For product states: $\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}|\psi\rangle^{(\mathrm{R})}, H_1] = 2a^2\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}, J_z^{(\mathrm{L})}]$

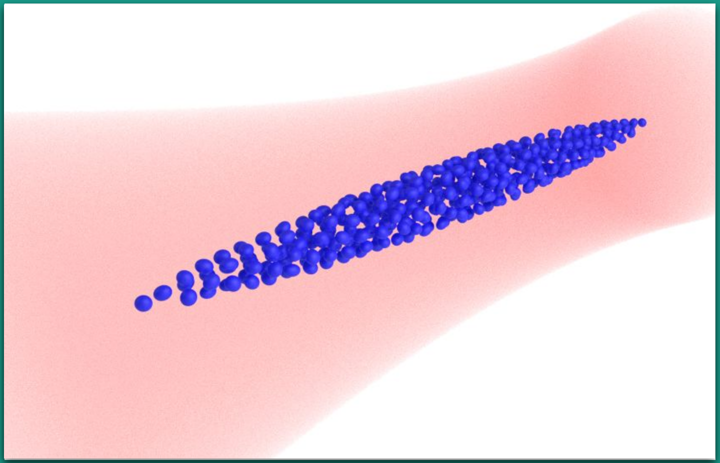
States	$\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{L})}, J_z^{(\mathrm{L})}]$	$(\Delta b_1)^{-2}\|_{\max}$
$\|j\rangle_y^{\otimes N_{\mathrm{L}}} \otimes \|j\rangle_y^{\otimes N_{\mathrm{L}}}$	$2N_{\mathrm{L}}j$	$2a^2Nj$
$\|\Psi_{\mathrm{sep}}\rangle \otimes \|\Psi_{\mathrm{sep}}\rangle$	$4N_{\mathrm{L}}j^2$	$4a^2Nj^2$
$\|\mathrm{GHZ}\rangle \otimes \|\mathrm{GHZ}\rangle$	N_{L}^2	$a^2N^2/2$
$\|\mathrm{D}_{N_{\mathrm{L}}}\rangle_x \otimes \|\mathrm{D}_{N_{\mathrm{L}}}\rangle_x$	$N_{\mathrm{L}}(N_{\mathrm{L}}+2)/2$	$a^2N(N+4)/4$



One-Dimensional Ensemble of Atoms

We now show bounds for an arbitrary **PERMUTATIONALLY INVARIANT** (PI) probability distribution function.

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_k \Pi_k [P(\boldsymbol{x})]$$



Precision bounds for a single ensemble

For states **INSENSITIVE** to the homogeneous fields:

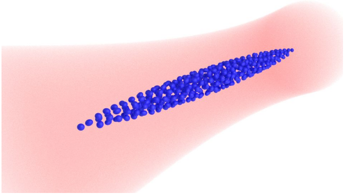
$$(\Delta b_1)^{-2}|_{\max} = (\sigma^2 - \eta) \sum_{n=1}^{N} \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}]$$

For states **SENSITIVE** to the homogeneous fields:

$$(\Delta b_1)^{-2}|_{\max} = (\sigma^2 - \eta) \sum_{n=1}^{N} \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z^{(n)}] + \eta \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, J_z]$$

Correlation between particle positions:

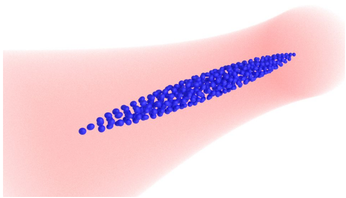
$$\frac{-\sigma^2}{N-1} \leqslant \eta \leqslant \sigma^2$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$
- Singlet: $\varrho_{\rm singlet}^{(\rm s)} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$
- Best separable: $|\psi_{\rm sep}\rangle = \left(\frac{|-j\rangle+|+j\rangle}{\sqrt{2}}\right)^{\otimes N}$
- $|{\rm GHZ}\rangle = \frac{|00\cdots 00\rangle + |11\cdots 11\rangle}{\sqrt{2}}$
- Unpolarized Dicke state (*x* and *z*):

$$|{\rm D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k(|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\mathrm{tp}}\rangle = |j\rangle_y^{\otimes N}$

$$(\Delta b_1)_{\mathrm{tp}}^{-2}|_{\max} = 2\sigma^2 N j$$

- Best separable: $|\psi_{\mathrm{sep}}\rangle = \left(\frac{|-j\rangle + |+j\rangle}{\sqrt{2}}\right)^{\otimes N}$

$$(\Delta b_1)_{\mathrm{sep}}^{-2}|_{\max} = 4\sigma^2 N j^2$$

- Unpolarized Dicke state (*x* and *z*):

$$|\mathrm{D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k(|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$

$$(\Delta b_1)_{\mathrm{D}}^{-2}|_{\max} = (\sigma^2 - \eta)N$$

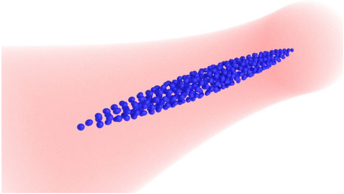
$$(\Delta b_1)_{\mathrm{D},x}^{-2}|_{\max} = (\sigma^2 - \eta)N + \eta\frac{N(N+2)}{2}$$

- Singlet: $\varrho_{\mathrm{singlet}}^{(\mathrm{s})} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$

$$(\Delta b_1)_{\mathrm{singlet}}^{-2}|_{\max} = (\sigma^2 - \eta)N\frac{4j(j+1)}{3}$$

- $|\mathrm{GHZ}\rangle = \frac{|00\cdots00\rangle + |11\cdots11\rangle}{\sqrt{2}}$

$$(\Delta b_1)_{\mathrm{GHZ}}^{-2}|_{\max} = (\sigma^2 - \eta)N + \eta N^2$$



Bounds for different spin states

- Totally polarized state: $|\psi_{\rm tp}\rangle = |j\rangle_y^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm tp}|_{\max} = 2\sigma^2 N j$$

- Best separable: $|\psi_{\rm sep}\rangle = \left(\frac{|-j\rangle + |+j\rangle}{\sqrt{2}}\right)^{\otimes N}$

$$(\Delta b_1)^{-2}_{\rm sep}|_{\max} = 4\sigma^2 N j^2$$

- Unpolarized Dicke state (*x* and *z*):

$$|{\rm D}_N\rangle_l = \binom{N}{N/2}^{-1/2} \sum_k \mathcal{P}_k(|0\rangle_l^{\otimes N/2} \otimes |1\rangle_l^{\otimes N/2})$$

$$(\Delta b_1)^{-2}_{\rm D}|_{\max} = (\sigma^2 - \eta)N$$

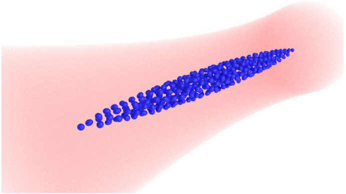
$$(\Delta b_1)^{-2}_{{\rm D},x}|_{\max} = (\sigma^2 - \eta)N + \eta\frac{N(N+2)}{2}$$

- Singlet: $\varrho^{(\rm s)}_{\rm singlet} = \sum_{D=1}^{D_0} p_D |0,0,D\rangle\langle 0,0,D|$

$$(\Delta b_1)^{-2}_{\rm singlet}|_{\max} = (\sigma^2 - \eta)N\frac{4j(j+1)}{3}$$

- $|{\rm GHZ}\rangle = \frac{|00\cdots 00\rangle + |11\cdots 11\rangle}{\sqrt{2}}$

$$(\Delta b_1)^{-2}_{\rm GHZ}|_{\max} = (\sigma^2 - \eta)N + \eta N^2$$



Conclusions

- We obtained **GENERAL FORMULAS** to compute the precision bounds for gradient magnetometry for spin-chains, double-wells, atomic single clouds and BECs.

- These bounds are based on the **INTERNAL STATE** of the system.

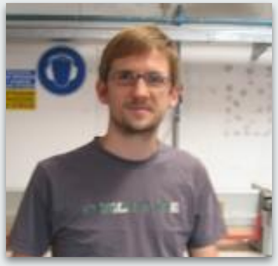
- Among the bounds we presented for an atomic cloud, there is the bound for the **BEST SEPARABLE STATE**.

- We found that all bounds appearing on this work are **SATURABLE**.

Thank you for your attention!

Iñigo Urizar-Lanz

Zoltán Zimborás



Philipp Hyllus

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Please, for more information, visit our preprint at arXiv.org:1703.09056

Simultaneous measurements

Condition for simultaneous measurements:

$$[L(\varrho, H_0), L(\varrho, H_1)] = 0$$

Symmetric logarithmic derivative (SLD):

$$L(\varrho, H_0) = \mathbf{1}^{(\mathrm{x})} \otimes L(\varrho^{(s)}, J_z)$$
$$L(\varrho, H_1) = \sum_{n=1}^{N} \int \mathrm{d}\boldsymbol{x}\, x_n |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \otimes L(\varrho^{(s)}, j_z^{(n)})$$

Example: PI states

$$L(\varrho, H_1) = \hat{\mu}^{(\mathrm{x})} \otimes L(\varrho^{(s)}, J_z)$$