k-stretchability of entanglement, and the duality of k-separability and k-producibility

Seminar of the Department of Theoretical Physics, UPV/EHU, Bilbao

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MOMENTUM OF INNOVATION

Introduction

Bipartite correlation and entanglement

- classification/qualification/quantification: (S)LO(CC)
- uncorrelated/correlated, and separable/entangled

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Permutation invariant properties

- three-level structure, Young-diagrams
- k-partitionability (k-separability), k-producibility (ent. depth), duality
- k-stretchability

Quantum states

States of discrete finite quantum systems

- *state vector:* $|\psi\rangle \in \mathcal{H}$ (normalized) superposition
- *pure state:* $\pi = |\psi\rangle\langle\psi| \in \mathcal{P}$
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- the decomposition is not unique

Mixedness and distinguishability

Measure of mixedness:

- von Neumann entropy: $S(\varrho) = -\operatorname{Tr} \varrho \ln \varrho$
- concave, nonnegative, vanishes iff ϱ pure
- Schur-concavity: *entropy* = *mixedness*
- increasing in bistochastic quantum channels
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Measure of distinguishability:

- (Umegaki's) quantum relative entropy: $D(\varrho||\sigma) = \operatorname{Tr}\varrho(\ln\varrho - \ln\sigma)$
- jointly convex, nonnegative, vanishes iff $\varrho = \omega$
- quantum Stein's lemma: *relative entropy* = *distinguishability*

  (rate of decaying of the probability of error in hypothesis testing, Hiai & Petz)
- decreasing in quantum channels

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Notions of correlation:

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- in q.m. there are many (nontrivially) different observables in a system
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- the state is *uncorrelated* iff $\mathrm{COV}(A,B) = 0$ for all A, B,
  iff $\langle AB \rangle = \langle A \rangle \langle B \rangle$ for all A, B, iff $\varrho = \varrho_1 \otimes \varrho_2$, iff $\Gamma = 0$

Bipartite correlation and entanglement

Pure states

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- state of subsystem (e.g., $\operatorname{Tr}_2 \pi \in \mathcal{D}_1$) not necessarily pure
- π is entangled if (and only if) $\operatorname{Tr}_2 \pi$ and $\operatorname{Tr}_1 \pi$ are mixed
  In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)

Bipartite correlation and entanglement

Mixed states: correlation

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- classically correlated sources produce states of this kind (Werner)
  preparable by Local Operations and Classical Communication (LOCC),
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  preparable by Local Operations and Classical Communication (LOCC),
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- the decomposition is not unique
- deciding separability is difficult

Bipartite correlation and entanglement – measures

- correlation "of the state itself": $\Gamma := \varrho - \varrho_1 \otimes \varrho_2$
  then $\mathrm{COV}(\varrho; A, B) = \langle AB \rangle - \langle A \rangle \langle B \rangle = \mathrm{Tr}\,\Gamma A \otimes B = \langle \Gamma | A \otimes B \rangle_{\mathrm{HS}}$
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- correlation might not be seen well from COV, but for all A, B,

$$\frac{1}{2}\mathrm{COV}(\varrho; \hat{A}, \hat{B})^2 \leq C(\varrho), \qquad \hat{A} = A/\|A\|_\infty, \hat{B} = B/\|B\|_\infty$$

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- faithful: $C(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\text{unc}}$, $E(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\text{sep}}$
- $E(\varrho)$ is hard to calculate

Multipartite correlation and entanglement – structure

Level 0.: subsystems — Boolean lattice structure: $P_0 = 2^L$

- whole system: $L = \{1, 2, \dots, n\}$
- subsystem: $X \subseteq L$, then $\mathcal{H}_X$, $\mathcal{P}_X$, $\mathcal{D}_X$

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)
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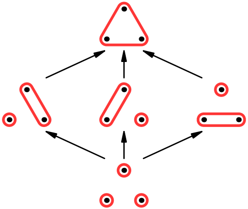

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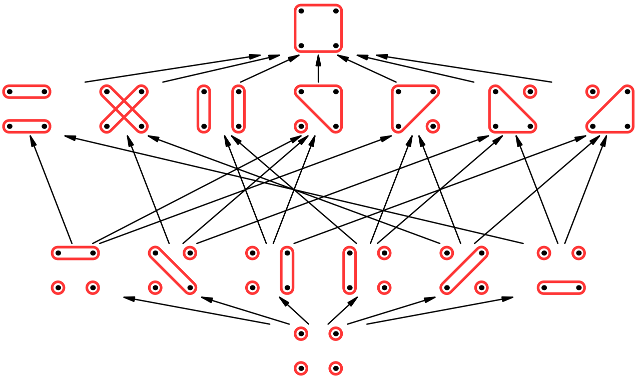

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  LO-closed — $\upsilon \preceq \xi \Leftrightarrow \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}$
- *ξ-separable states:* $\mathcal{D}_{\xi\text{-sep}} = \operatorname{Conv} \mathcal{D}_{\xi\text{-unc}}$
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$$C_\xi(\varrho) = \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D(\varrho||\sigma) = \sum\nolimits_{X\in\xi} S(\varrho_X) - S(\varrho)$$

LO-monotone (proper correlation measure)

- *ξ-entanglement (of formation):*

$$E_\xi(\pi) = C_\xi|_{\mathcal{P}}(\pi), \qquad E_\xi(\varrho) = \min\Big\{\sum\nolimits_i p_i E_\xi(\pi_i)\;\Big|\;\sum\nolimits_i p_i\pi_i = \varrho\Big\}$$

LOCC-monotone (proper entanglement measure)

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Multipartite correlation and entanglement – measures

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Multipartite correlation and entanglement – structure

Level II.: multiple partitions   lattice structure: $P_{\mathrm{II}} = \mathcal{O}_{\downarrow}(P_{\mathrm{I}}) \setminus \{\emptyset\}$

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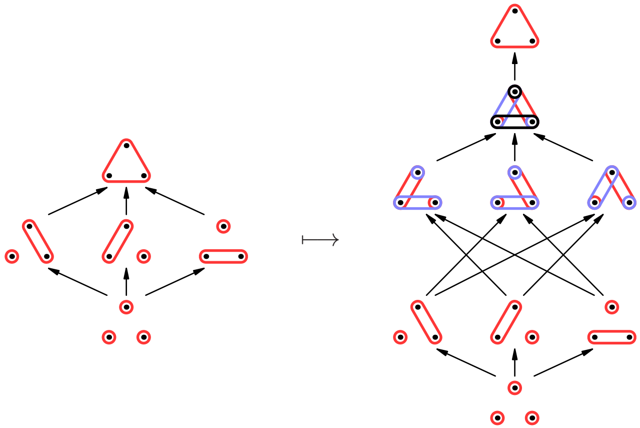

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$$\boldsymbol{\mu}_k = \big\{\mu \in P_{\mathrm{I}} \;\big|\; |\mu| \geq k\big\}, \qquad \boldsymbol{\nu}_k = \big\{\nu \in P_{\mathrm{I}} \;\big|\; \forall N \in \nu : |N| \leq k\big\}$$

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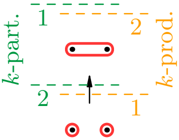

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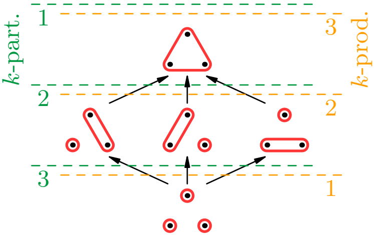

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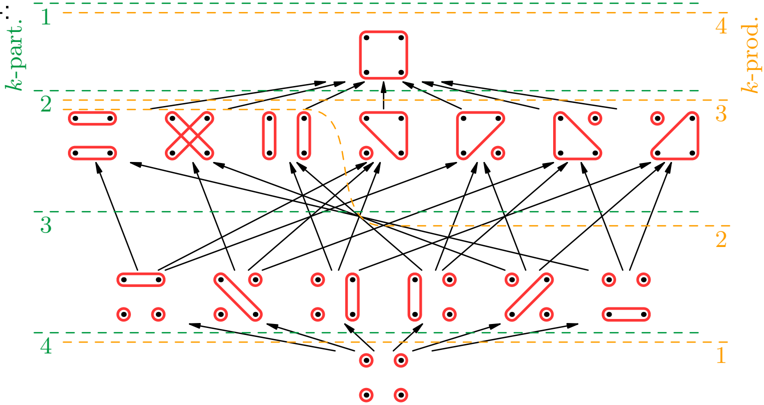

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- with these:
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  k-partitionably and *k-producibly separable* states

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  k-partitionability and *k-producibility entanglement*

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Example: Electron system of molecules

- elementary subsystems: localized atomic orbitals (Pipek-Mezey)
- "atomic split": $\alpha = \{A_1, A_2, \dots, A_{|\alpha|}\}$ (blue)
- "bond split": $\beta = \{B_1, B_2, \dots, B_{|\beta|}\}$ (red)

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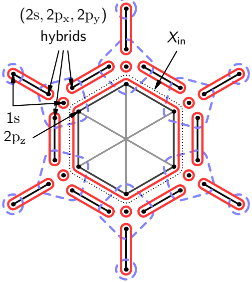
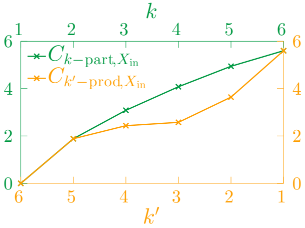
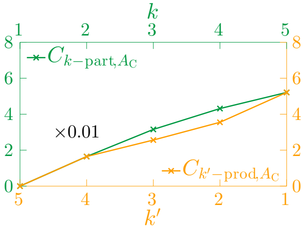
benzene ($\mathrm{C_6H_6}$): $C_\alpha = 29.52$, $C_\beta = 2.33$







(in units ln 4)

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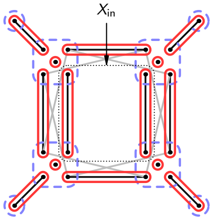
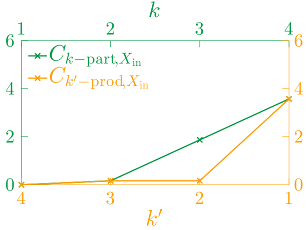
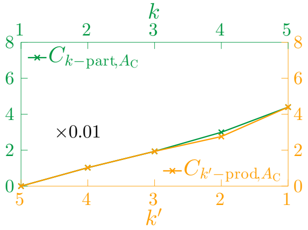
cyclobutadiene (C_4H_4): $C_\alpha = 19.48$, $C_\beta = 3.17$







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Entanglement classes

Level III: Entanglement classes lattice structure: $P_{\text{III}} = \mathcal{O}_{\uparrow}(P_{\text{II}}) \setminus \{\emptyset\}$

- ideal filter: $\underline{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{|\underline{\boldsymbol{\xi}}|}\} \subseteq P_{\text{II}}$ (closed upwards w.r.t. $\preceq$)
- partial order: $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$ def.: $\underline{\boldsymbol{\upsilon}} \subseteq \underline{\boldsymbol{\xi}}$

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Entanglement classes

Level III: Entanglem

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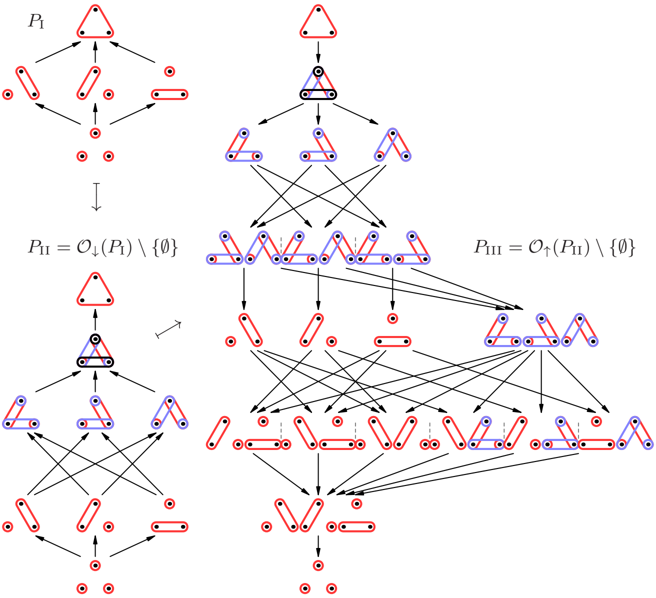
- partial order: $\underline{\upsilon}$



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$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}} := \bigcap_{\boldsymbol{\xi} \notin \underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} \cap \bigcap_{\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$$

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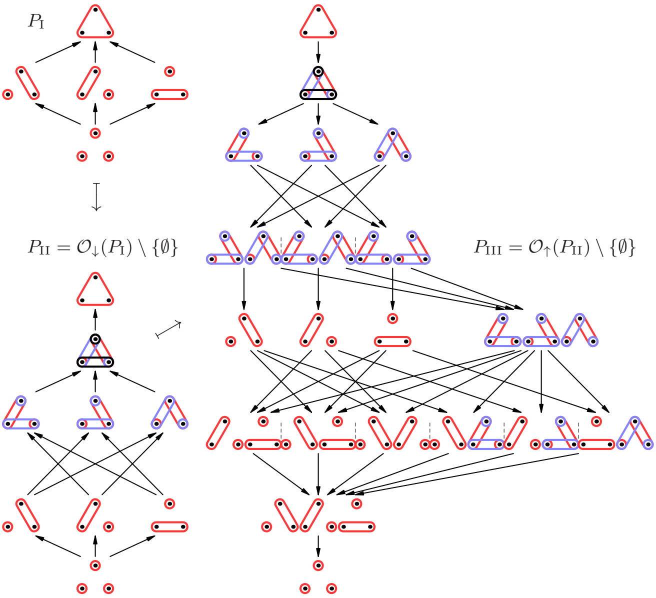

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- LOCC convertibility:
  if $\exists \varrho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}}$, $\exists \Lambda$ LOCC map s.t. $\Lambda(\varrho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$

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Entanglement classes

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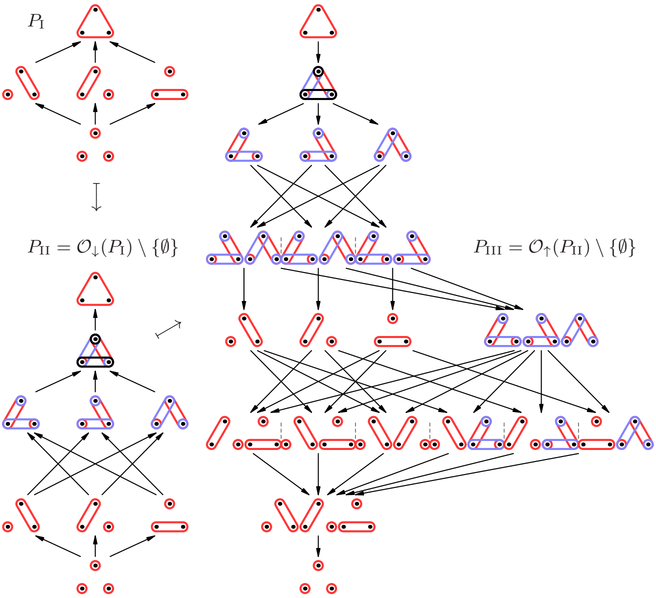

Correlation classes

Level III: Corr./Ent. classes   lattice structure: $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$

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Szalay, PRA **92**, 042329 (2015)

Correlation classes

Level III: Corr./Ent. classes   lattice structure: $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$

- partial correlation classes: intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$

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proven constructively for $n = 3$ Han, Kye, PRA **99**, 032304 (2019)

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Szalay, JPhysA **51**, 485302 (2018)

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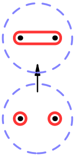
Permutation symmetric correlation and entanglement

Level I.: splitting type of the system of n elementary subsystems

$n = 2$:



Szalay, arXiv:1906.10798 [quant-ph]

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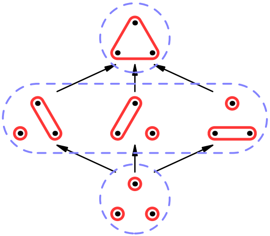

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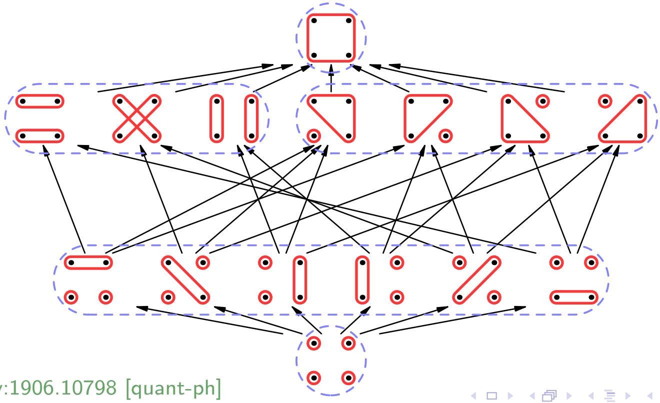

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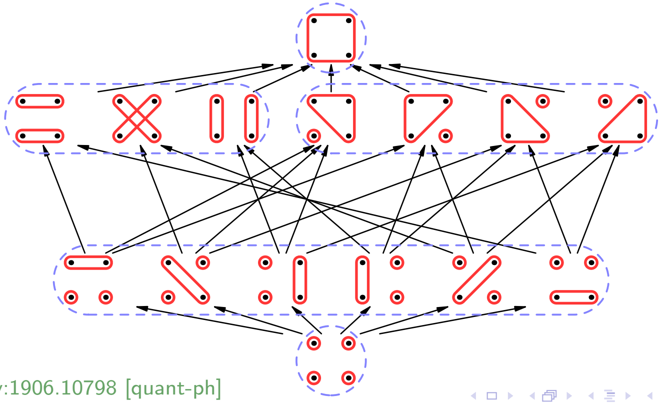

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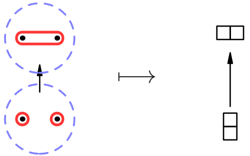

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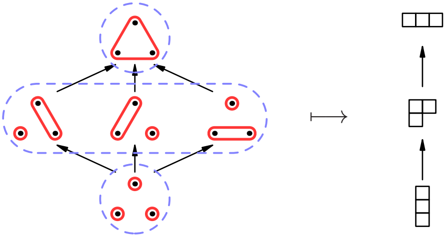

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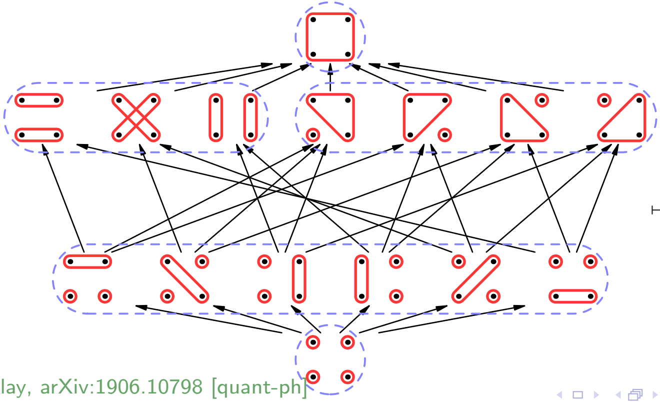
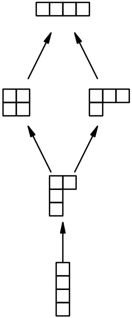


$\longmapsto$



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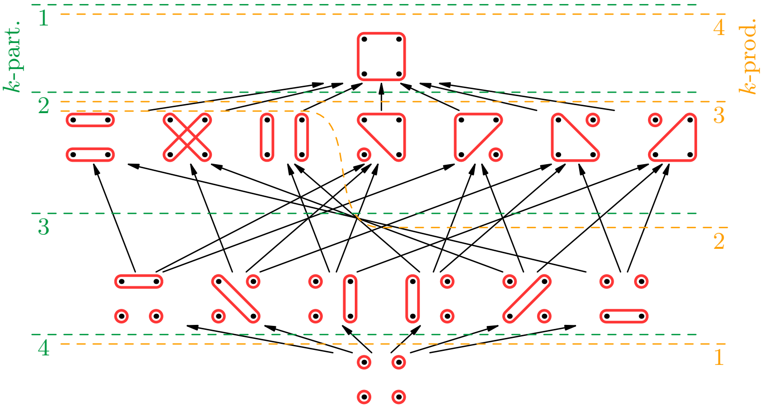
Permutation symmetric correlation and entanglement

Structure of k-partitionability and k-producibility

- P_I graded lattice, gradation = partitionability



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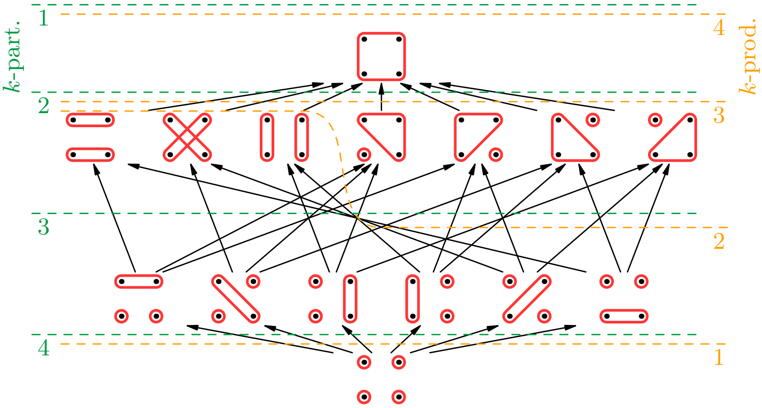

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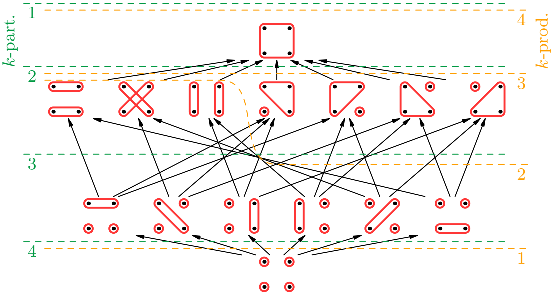

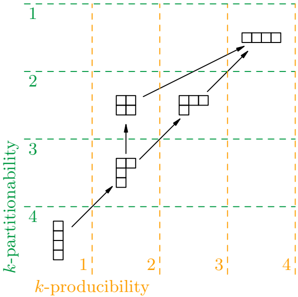

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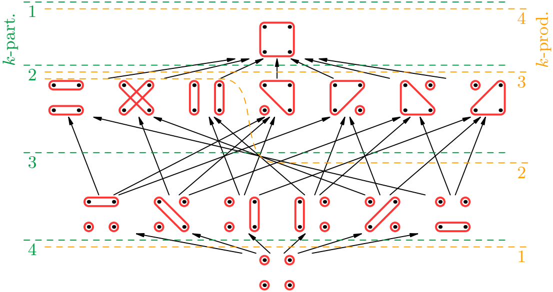

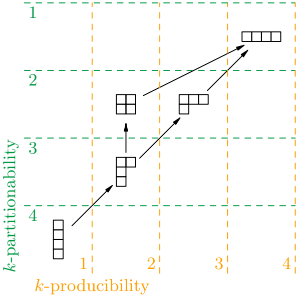

- note: $\preceq$ is not respected by the conjugation

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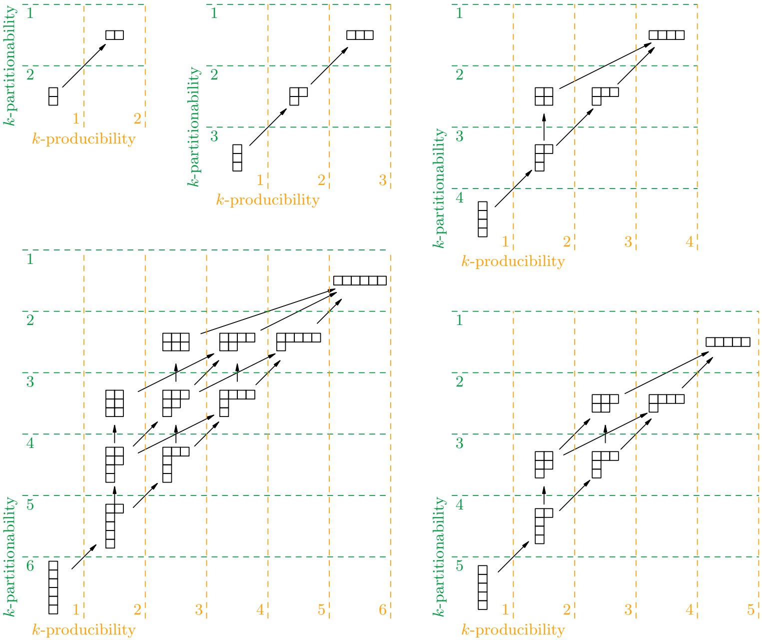
k-partitionability
k-producibility
1
2
3
4
5
6

Permutation symmetric correlation and entanglement

Construction

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- $s(X) := |X|$, and elementwisely on P_{I}, works also for P_{II} and P_{III}
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\begin{array}{ccc}
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height, width and rank of a Young diagram

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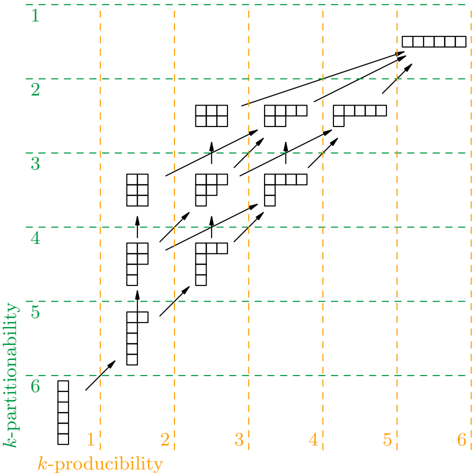

k-partitionability, k-producibility and k-stretchability

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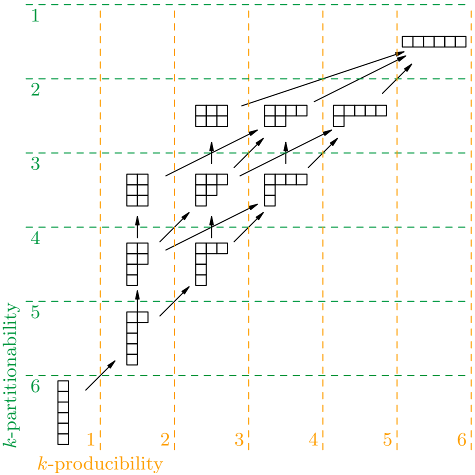

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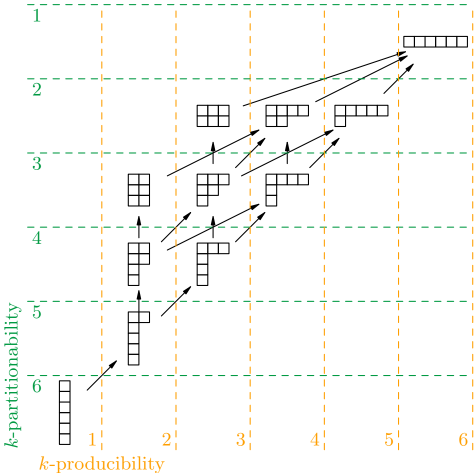

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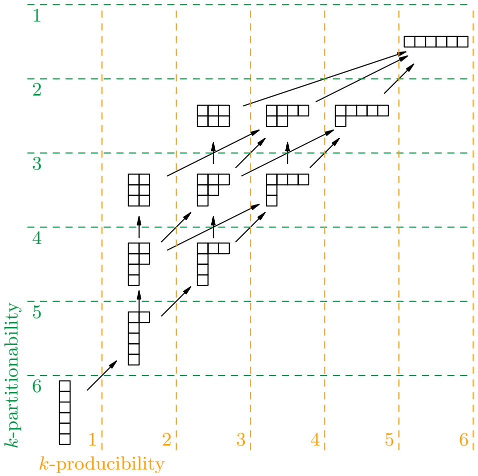

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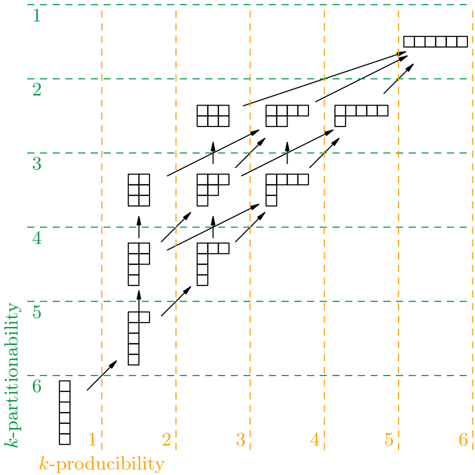

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$_l \preceq \hat{\boldsymbol{\nu}}_k, \quad \hat{\boldsymbol{\tau}}_l \preceq \hat{\boldsymbol{\tau}}_k \quad \Longleftrightarrow \quad l \leq k$

$_{+1-k}, \hat{\boldsymbol{\nu}}_k \preceq \hat{\boldsymbol{\mu}}_{\lceil n/k \rceil}$, from

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k-partitionability, k-producibility and k-stretchability

height, width and rank of a Young diagram $\Longrightarrow$ properties

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- monotones

$$\hat{\upsilon} \prec \hat{\xi} \quad\Longrightarrow\quad h(\hat{\upsilon}) > h(\hat{\xi}), \quad w(\hat{\upsilon}) \leq w(\hat{\xi}), \quad r(\hat{\upsilon}) < r(\hat{\xi}).$$

- chains

$$\hat{\boldsymbol{\mu}}_l \preceq \hat{\boldsymbol{\mu}}_k \quad\Longleftrightarrow\quad l \geq k \qquad\qquad \hat{\boldsymbol{\nu}}_l \preceq \hat{\boldsymbol{\nu}}_k, \quad \hat{\boldsymbol{\tau}}_l \preceq \hat{\boldsymbol{\tau}}_k \quad\Longleftrightarrow\quad l \leq k$$

- bounds among properties: $\hat{\boldsymbol{\mu}}_k \preceq \hat{\boldsymbol{\nu}}_{n+1-k}$, $\hat{\boldsymbol{\nu}}_k \preceq \hat{\boldsymbol{\mu}}_{\lceil n/k \rceil}$, from

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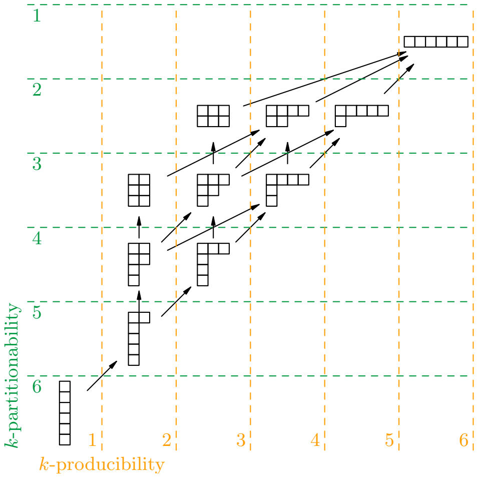

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Thank you for your attention!

Szalay, arXiv:1906.10798 [quant-ph] (submitted after the talk)

Szalay, JPhysA **51**, 485302 (2018)

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)

Szalay, PRA **92**, 042329 (2015)

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