Entanglement detection from randomized measurements

Satoya Imai @ University of Siegen

Quantum Glue Meetings in Bilbao: 16. 11. 2022

PRL (2021)$^{\otimes 2}$ & PRA (2022)

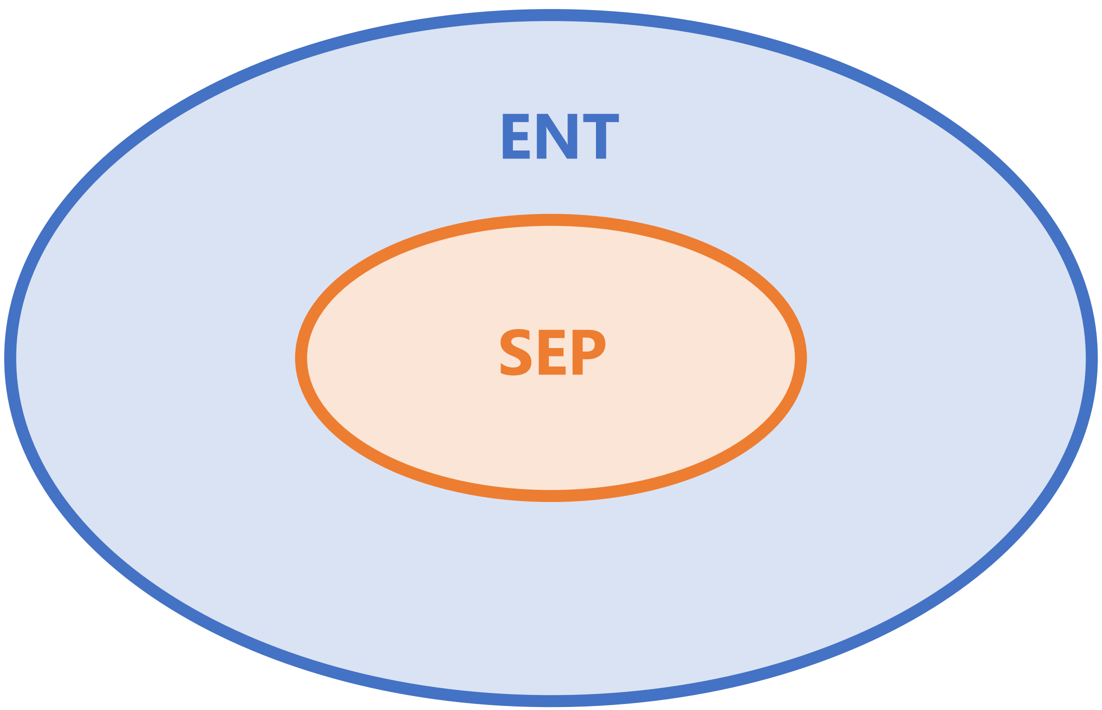
arXiv:2205.08447 & ongoing projects


Schleswig-
Holstein
Mecklenburg-
Vorpommern
Bremen
Hamburg
Lower Saxony
Brandenburg
Saxony-
Anhalt
North Rhine-
Westphalia
Siegen
Saxony
Hesse
Rhineland-
Palatinate
Baden-
Württemberg
Bavaria
Ontheworldmap.com

Outline

- Basics of entanglement
- Randomized measurements
- Conclusion

Basics of entanglement



Recap: Single-qudit state

Pure state

- d-dimensional complex vector $|\psi\rangle \in H_d$, s.t. $\langle\psi|\psi\rangle = 1$

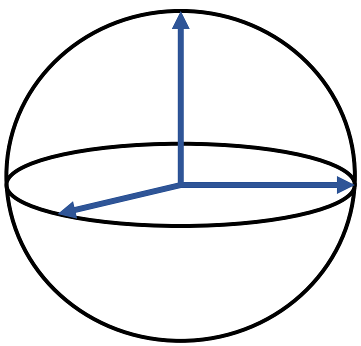
Mixed stat

- $d \times d$-matrix $\varrho \in H_d$, s.t. $\varrho^\dagger = \varrho$ and $\mathrm{tr}[\varrho] = 1$ and $\varrho \geq 0$

Single-qubit state: (d=2)

$$\varrho = \frac{1}{2}\left(I + \sum_{i=x,y,z} \langle\sigma_i\rangle_\varrho \, \sigma_i\right), \qquad \sum \langle\sigma_i\rangle_\varrho^2 \leq 1$$

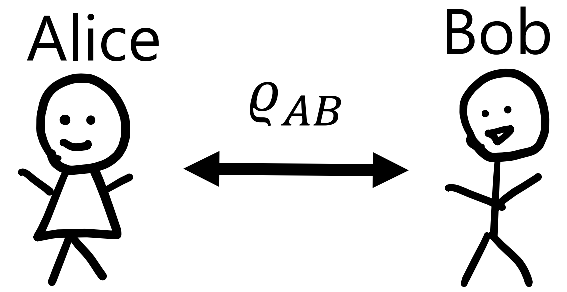
Bloch ball (d=2)



Bipartite entanglement

Pure state

- Product: $|\psi\rangle_{\mathrm{prod}} = |\phi\rangle_A \otimes |\phi\rangle_B$
  - $|0\rangle_A \otimes |0\rangle_B$
- Otherwise, entangled
  - $|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$



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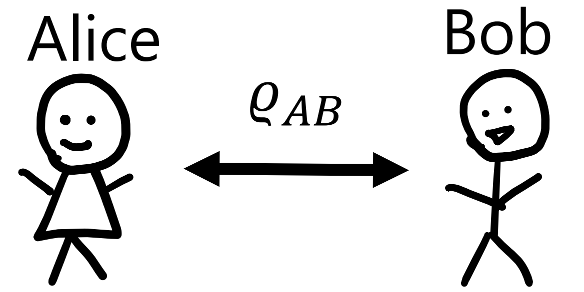
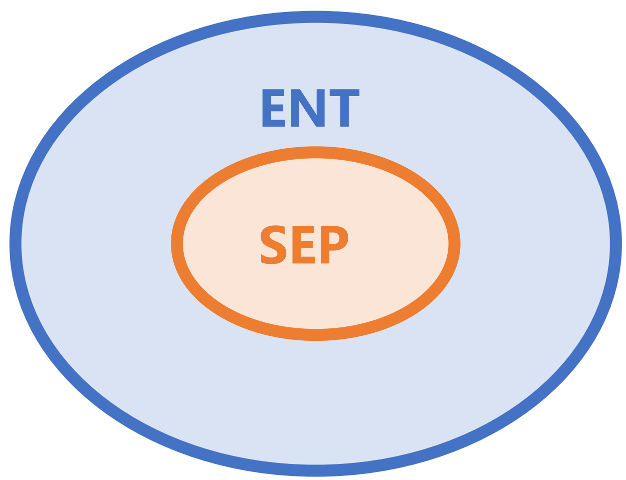
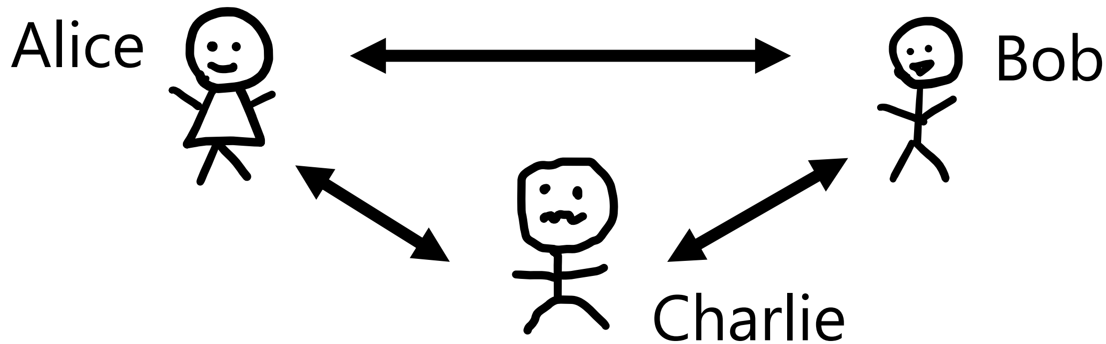


Mixed state

- Separable: $\varrho_{\text{sep}} = \sum p_i\, \rho_i^A \otimes \rho_i^B$
- Otherwise, entangled



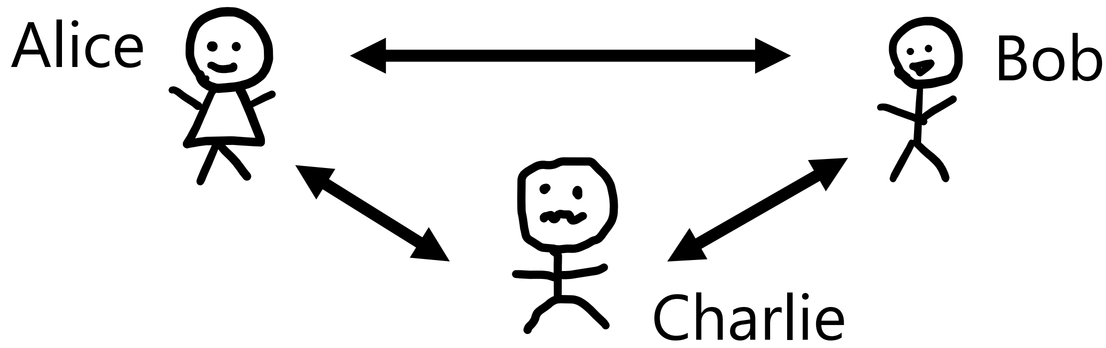
Multipartite entanglement



There are different forms

- Fully separable: $|\psi\rangle_{\mathrm{fs}} = |\phi\rangle_A \otimes |\phi\rangle_B \otimes |\phi\rangle_C$
  - $|0\rangle \otimes |0\rangle \otimes |0\rangle$

Multipartite entanglement



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- Biseparable: $|\psi\rangle_{\mathrm{bs}} = |\phi\rangle_{AB} \otimes |\phi\rangle_C$
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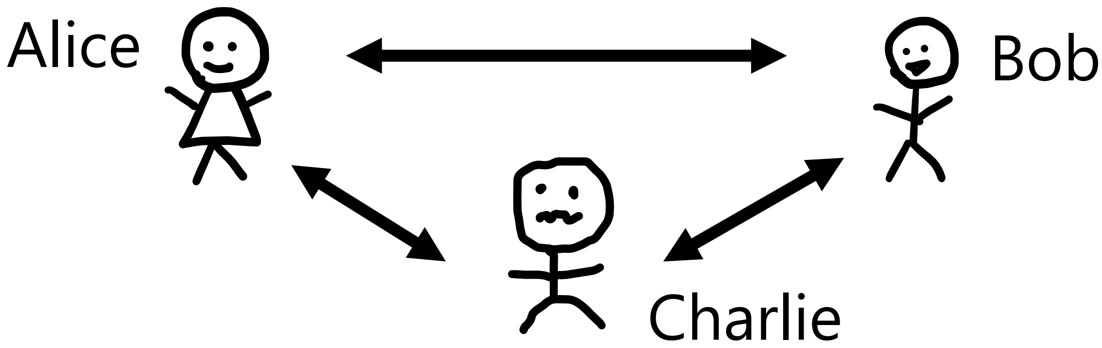
Multipartite entanglement



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  - Fully separable: $|\psi\rangle_{\mathrm{fs}} = |\phi\rangle_A \otimes |\phi\rangle_B \otimes |\phi\rangle_C$
    - $|0\rangle \otimes |0\rangle \otimes |0\rangle$
  - Biseparable: $|\psi\rangle_{\mathrm{bs}} = |\phi\rangle_{AB} \otimes |\phi\rangle_C$
    - $|\mathrm{Bell}\rangle \otimes |0\rangle$
  - Otherwise, genuine multiparticle entangled (GME)
    - $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ or $|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

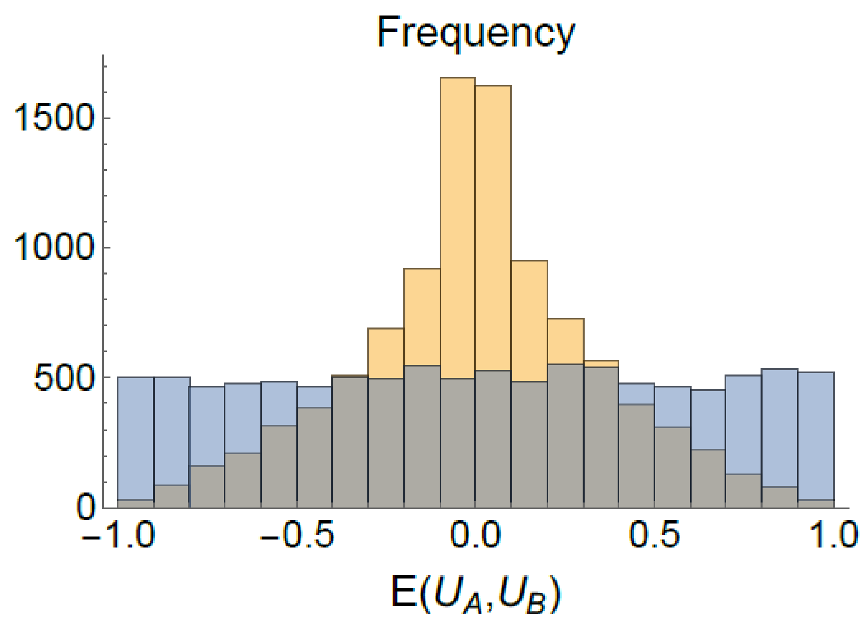
Separability problem

How to decide whether a given state is separable or not?

■ Importance

- Experimental generation of entanglement
- Fundamental difference from classical correlations
- Mathematical decomposition of matrices

Randomized Measurements

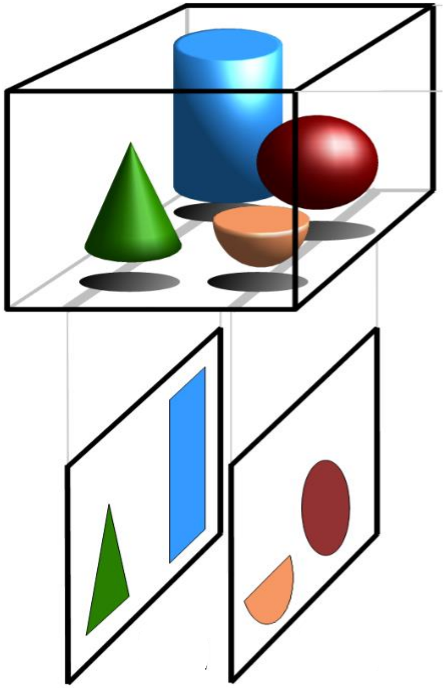


Our approach is different from shadow tomography*

*HY Huang, R Kueng, J Preskill, Nature Physics 2020

Why randomized measurements?

- **Several challenges**
  - State tomography
  -
  -



Why randomized measurements?

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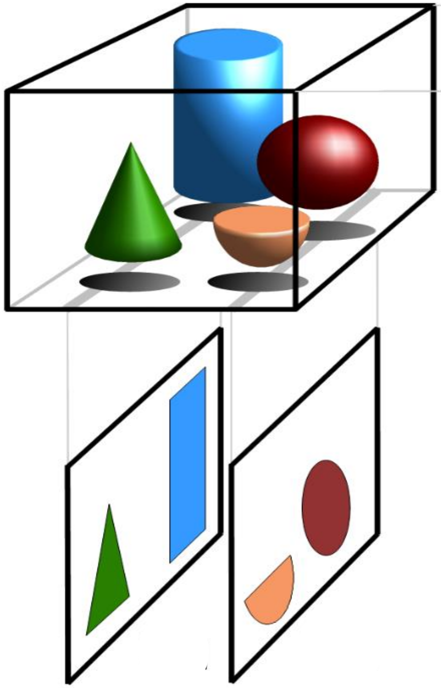
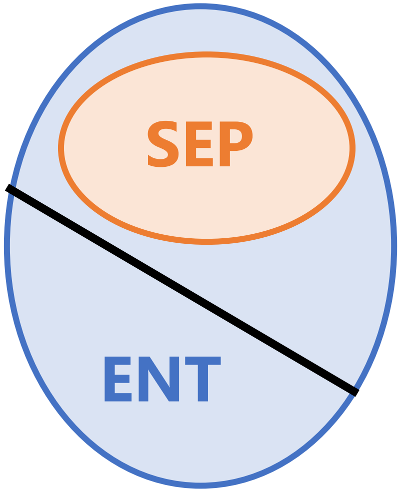
  -

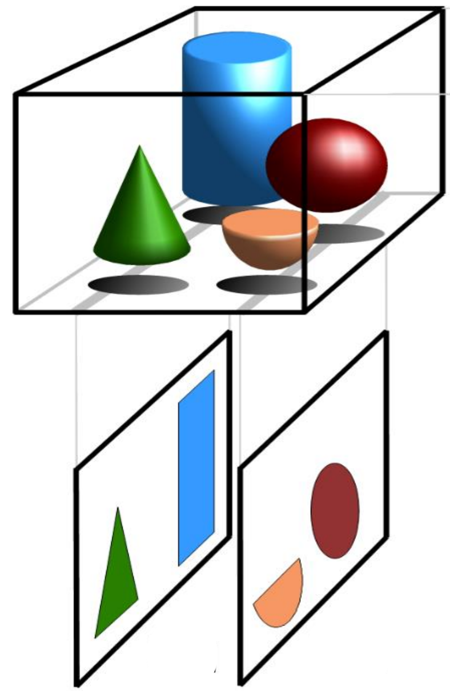


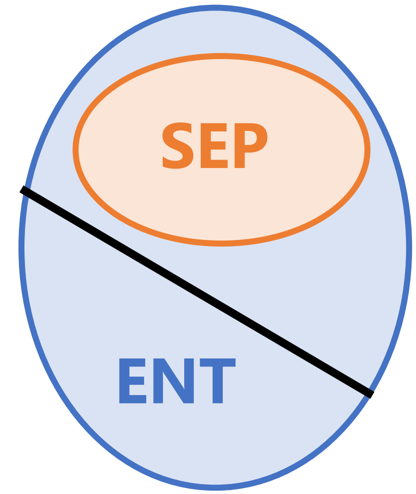


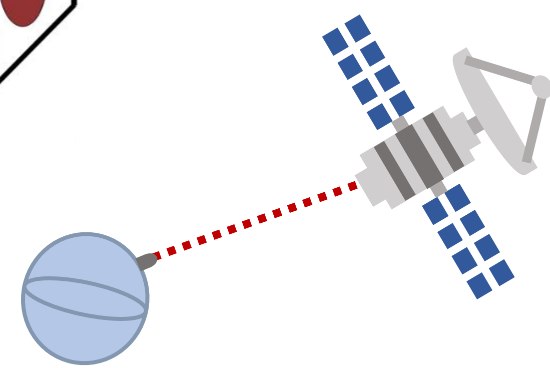
Why randomized measurements?

■ Several challenges

- State tomography
- Entanglement witness
- Sharing reference frame

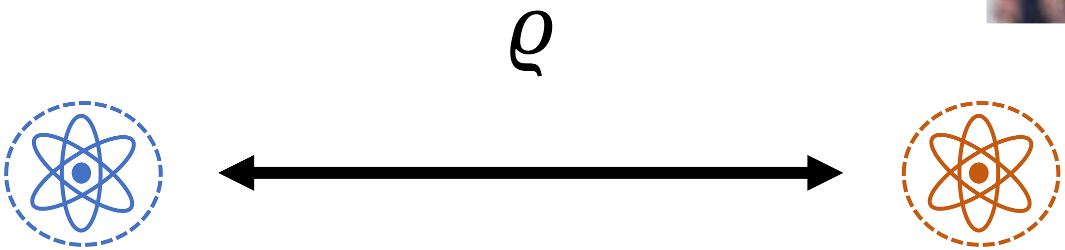






➡ **Randomized measurements!**

What are randomized measurements?



- They do not share common reference frame

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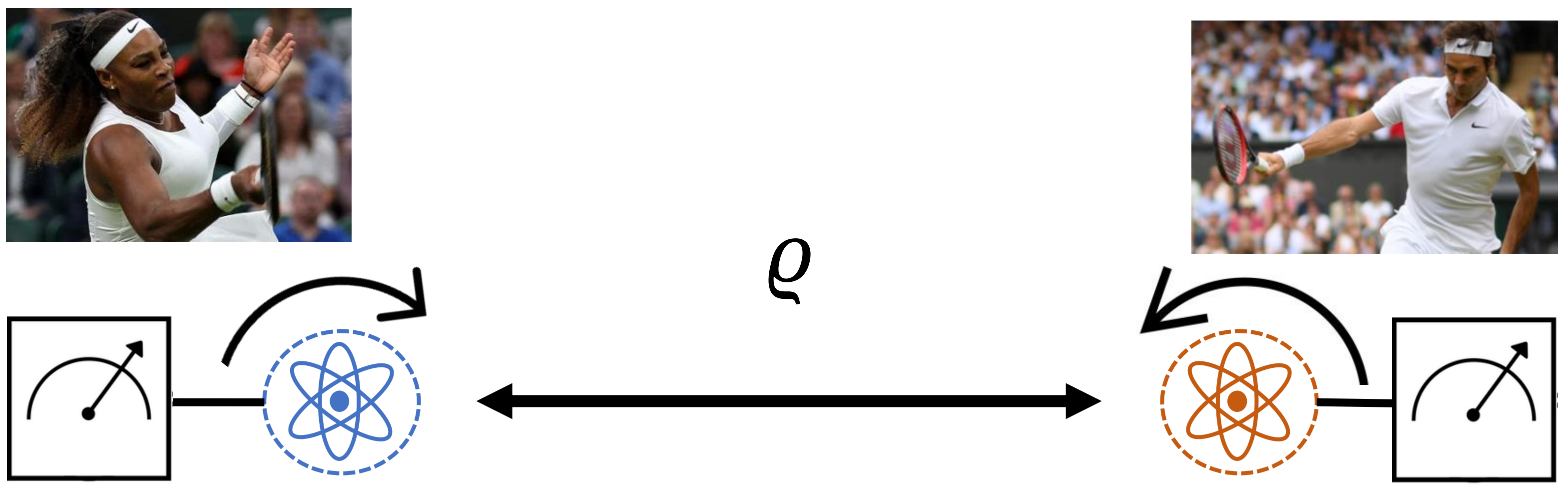
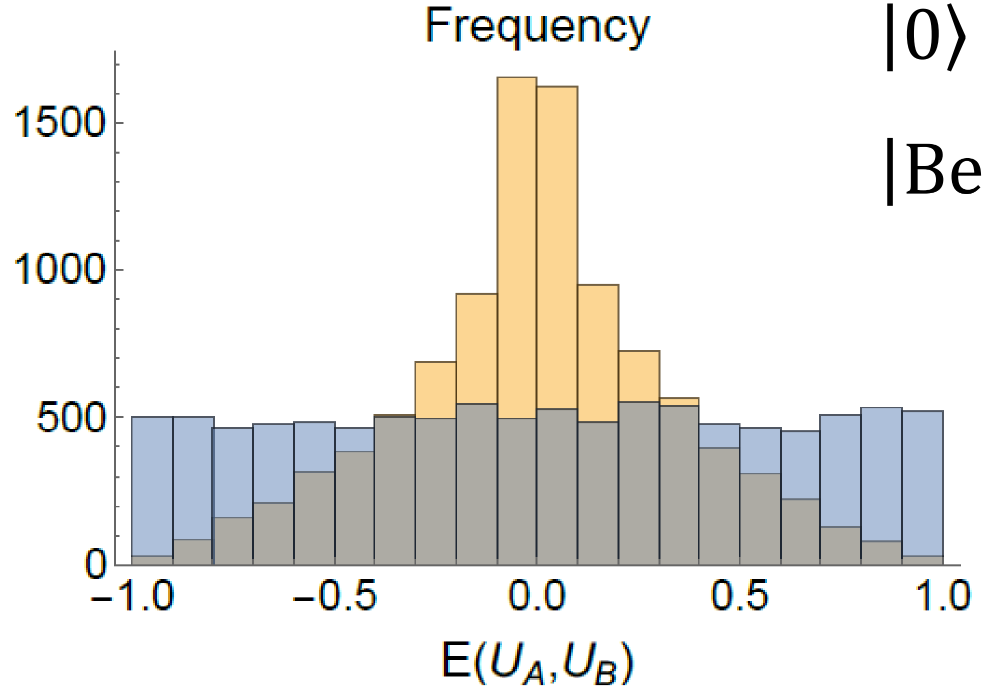


- They do not share common reference frame
- **Idea:** *Rotate* measurement direction arbitrarily

Random correlation function

$$E(U_A, U_B) = \mathrm{tr}\left(\varrho_{AB} U_A^\dagger \sigma_z U_A \otimes U_B^\dagger \sigma_z U_B\right)$$

Sample over random unitaries



$|0\rangle \otimes |0\rangle$

$|\mathrm{Bell}\rangle$

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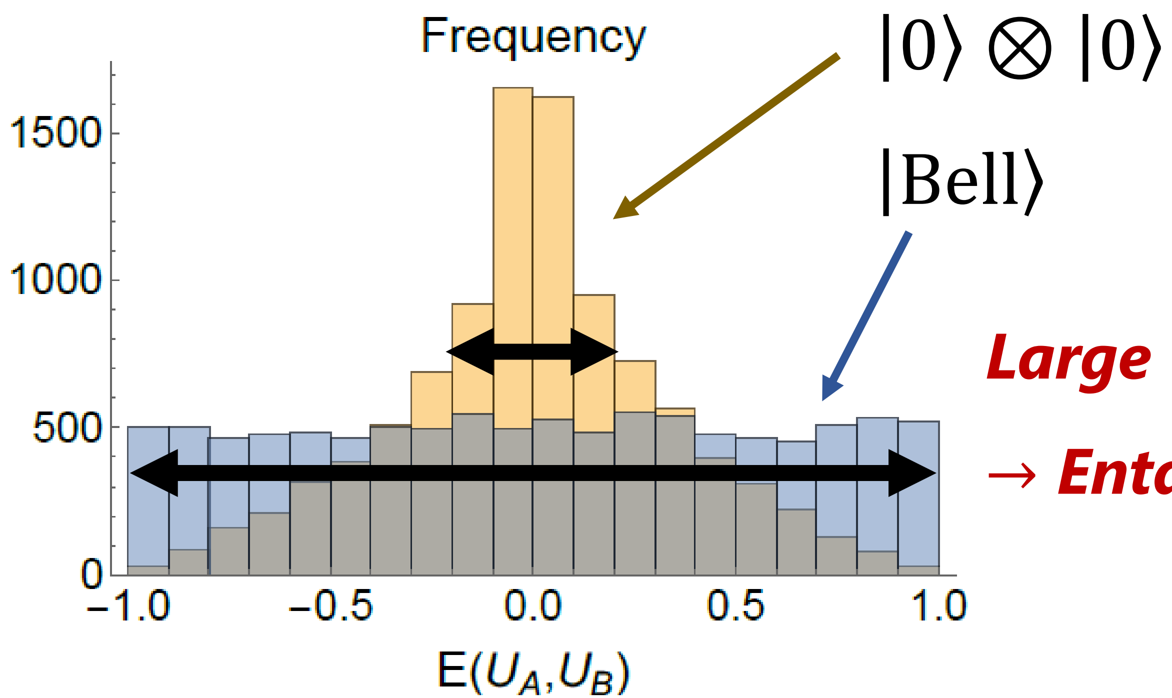


Large variance

→ Entangled?

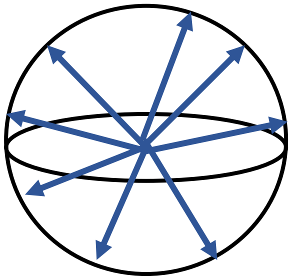
Moments over Haar unitaries

$$\mathcal{R}_{AB}^{(r)} = N \int dU_A \int dU_B \left[\mathrm{tr}\left(\varrho_{AB} U_A^\dagger \sigma_z U_A \otimes U_B^\dagger \sigma_z U_B\right)\right]^r$$

Simplifications: integral → sum

$$\mathcal{R}_{AB}^{(2)} \overset{!}{=} \sum_{i,j=x,y,z} \langle \sigma_i \otimes \sigma_j \rangle_{\varrho_{AB}}^2$$

Rotated Bloch vector



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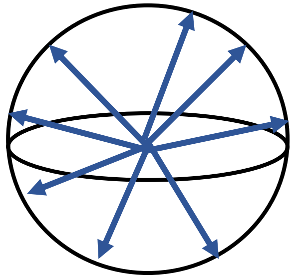

$$\mathcal{R}_{AB}^{(2)} \stackrel{!}{=} \sum_{i,j=x,y,z} \langle \sigma_i \otimes \sigma_j \rangle_{\varrho_{AB}}^2$$

Reference-frame-independent entanglement detection

$$\mathcal{R}_{AB}^{(2)} > 1 \Longrightarrow \varrho_{AB} \text{ is entangled}$$

MC Tran et al, PRA 2015

Geometry of two-qubit states

$$\mathcal{R}_{AB}^{(2)} = \sum \langle \sigma_i \otimes \sigma_j \rangle_{\varrho_{AB}}^2$$

$$\mathcal{R}_{A}^{(2)} + \mathcal{R}_{B}^{(2)} = \sum \langle \sigma_i \rangle_{\varrho_A}^2 + \sum \langle \sigma_i \rangle_{\varrho_B}^2$$

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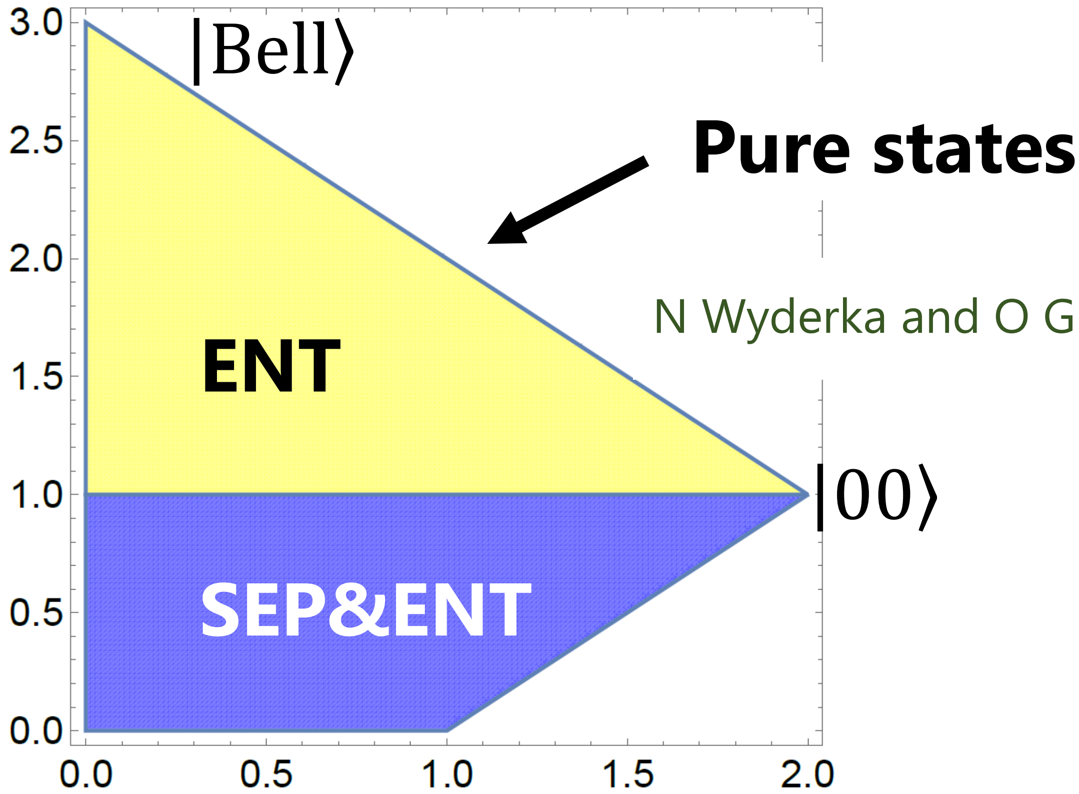

N Wyderka and O Gühne, JPA 2020

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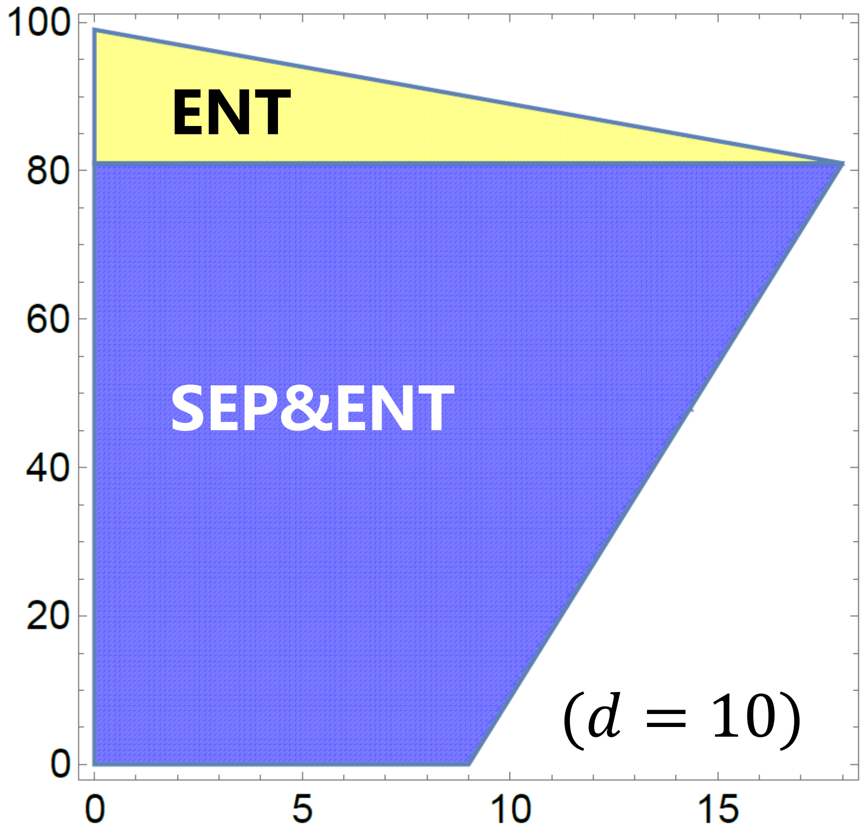
$\mathcal{R}_{AB}^{(2)} = \sum \langle \lambda_i \otimes \lambda_j \rangle_{\varrho_{AB}}^2$ λ_i: Gell-Mann matrices



$\mathcal{R}_{AB}^{(2)} > (d-1)^2 \Longrightarrow \mathrm{ENT}$

MC Tran et al, PRA 2016

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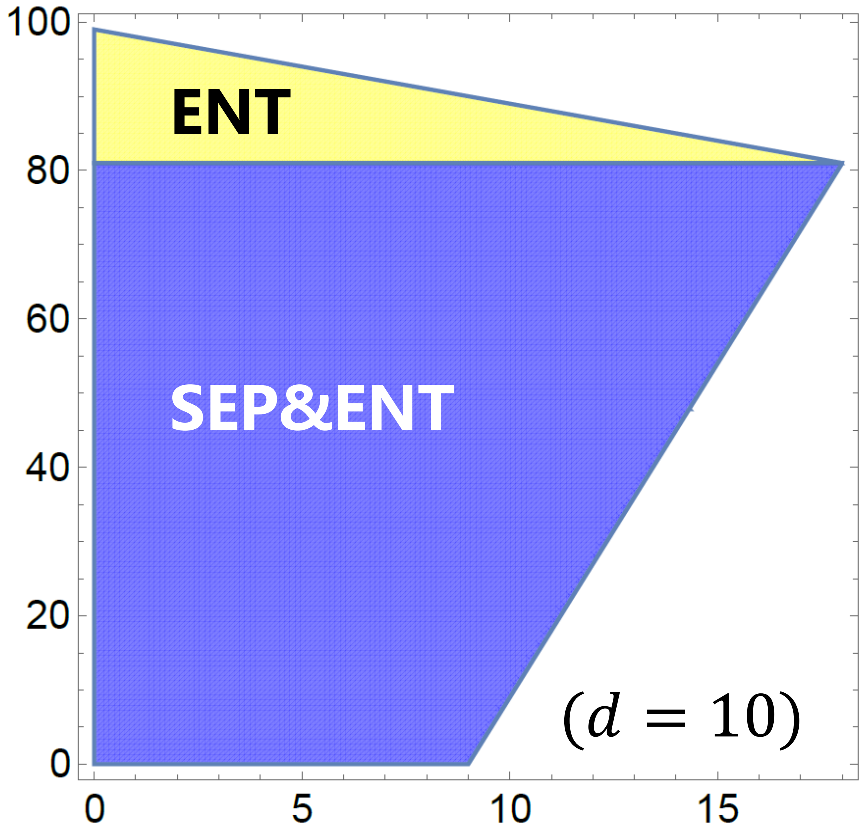

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MC Tran et al, PRA 2016

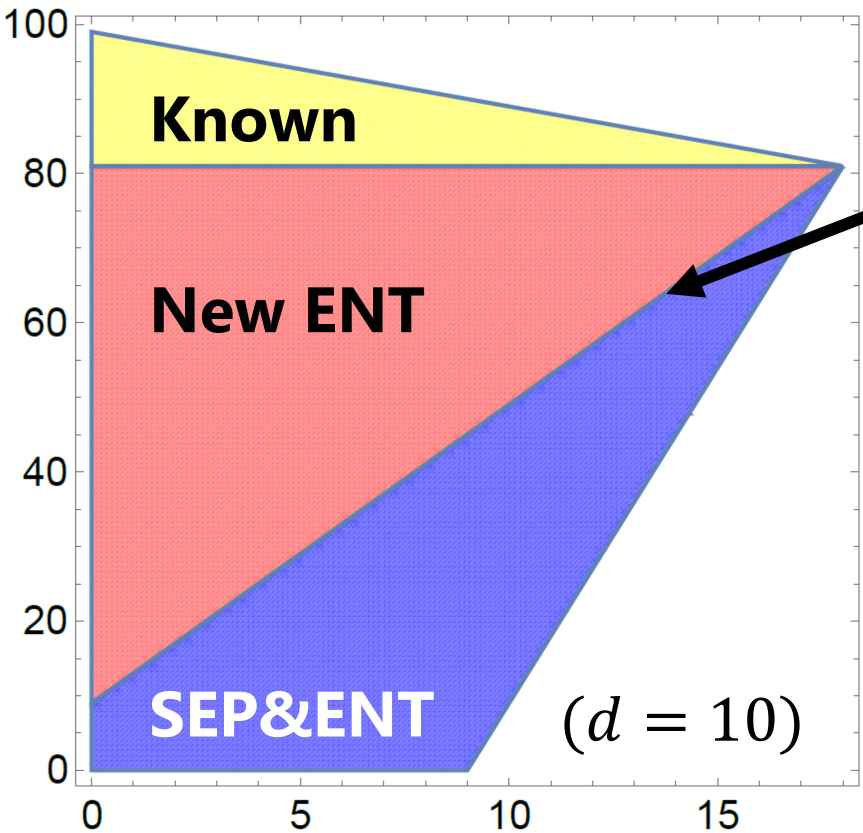
Questions: Can we detect

1. entanglement optimally?
2. Schmidt number?
3. NPT entanglement?
4. bound entanglement?

→ **We solved all of them!**

Result 1: Optimal separability criterion

$$\mathcal{R}_{AB}^{(2)} = \sum \langle \lambda_i \otimes \lambda_j \rangle_{\varrho_{AB}}^2$$



$$\mathcal{R}_{A}^{(2)} + \mathcal{R}_{B}^{(2)} = \sum \langle \lambda_i \rangle_{\varrho_A}^2 + \sum \langle \lambda_i \rangle_{\varrho_B}^2$$

Result 1: Optimal line, which is equivalent to

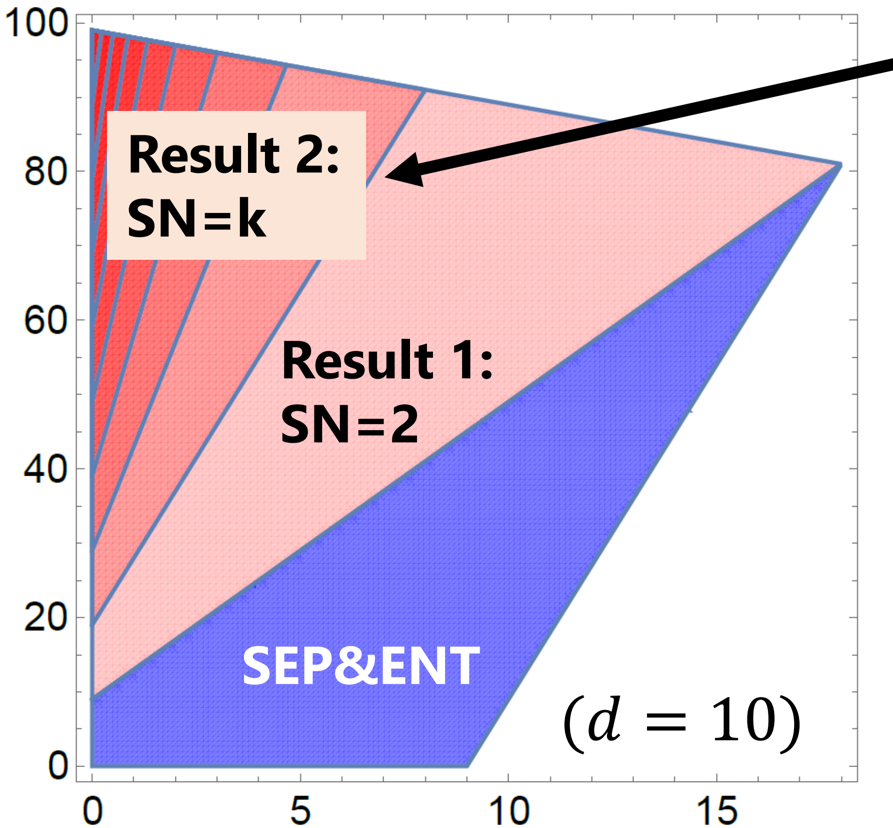
$$\mathrm{tr}[\varrho_{AB}^2] \leq \mathrm{tr}[\varrho_A^2]$$

R&P&M Horodecki, PLA 1996

SI, N Wyderka, A Ketterer, O Gühne, PRL 2021

Result 2: Schmidt number detection

$\mathcal{R}_{AB}^{(2)} = \sum \langle \lambda_i \otimes \lambda_j \rangle_{\varrho_{AB}}^2$

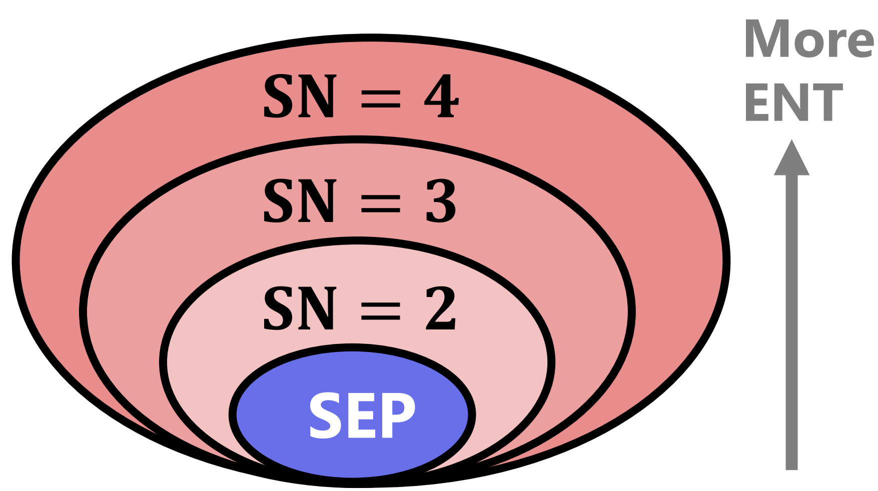


$\mathcal{R}_A^{(2)} + \mathcal{R}_B^{(2)} = \sum \langle \lambda_i \rangle_{\varrho_A}^2 + \sum \langle \lambda_i \rangle_{\varrho_B}^2$

Result 2: SN lines,

which are equivalent to

$$\mathrm{tr}[\varrho_{AB}^2] \leq k \, \mathrm{tr}[\varrho_A^2]$$



SI, O Gühne, S Nimmrichter
arXiv:2205.08447 (2022)

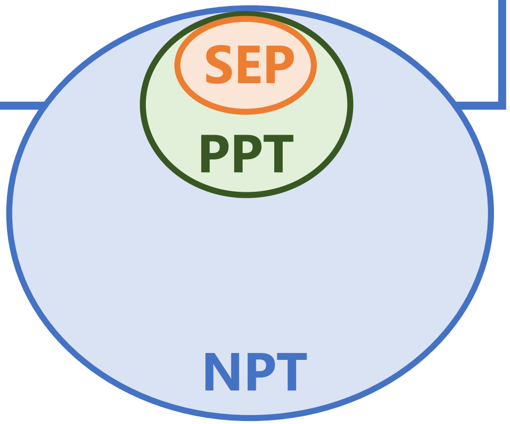
Result 3: Efficient NPT detection

- **Positive Partial Transpose (PPT) criterion:**

$$\varrho_{AB} \in \mathrm{SEP} \Longrightarrow \varrho_{AB}^{T_A} \geq 0 \text{ (called PPT state)}$$

A Peres PRL 1996; M&P&R Horodecki PLA 1996

- **Lesson:** PPT is outer approximation of SEP



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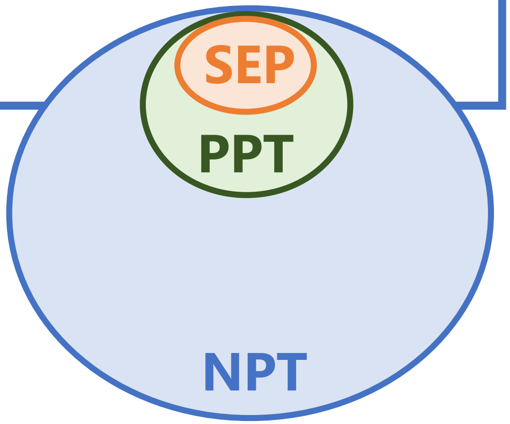

- **PT moments:** $p_k = \mathrm{tr}[(\varrho_{AB}^{T_A})^k]$

- Known criterion: A Elben et al, PRL 2020

$$\varrho_{AB} \in \mathrm{PPT} \Longrightarrow p_3 - p_2^2 \geq 0$$

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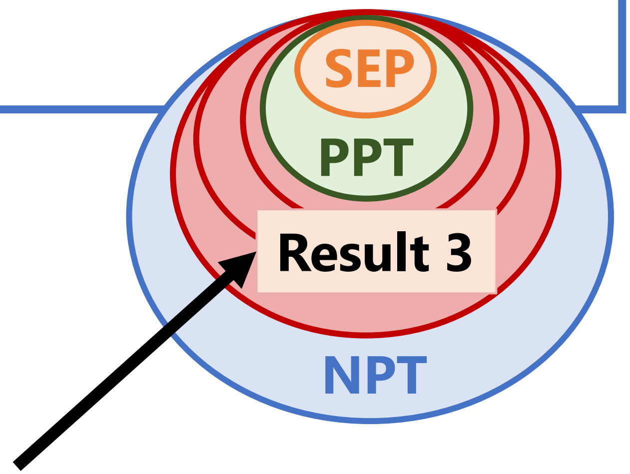

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$$\varrho_{AB} \in \text{PPT} \Longrightarrow p_3 - p_2^2 \geq 0$$

Result 3: Systematic methods to detect NPT efficiently

- **Idea:** Turn separability problem to positivity problem

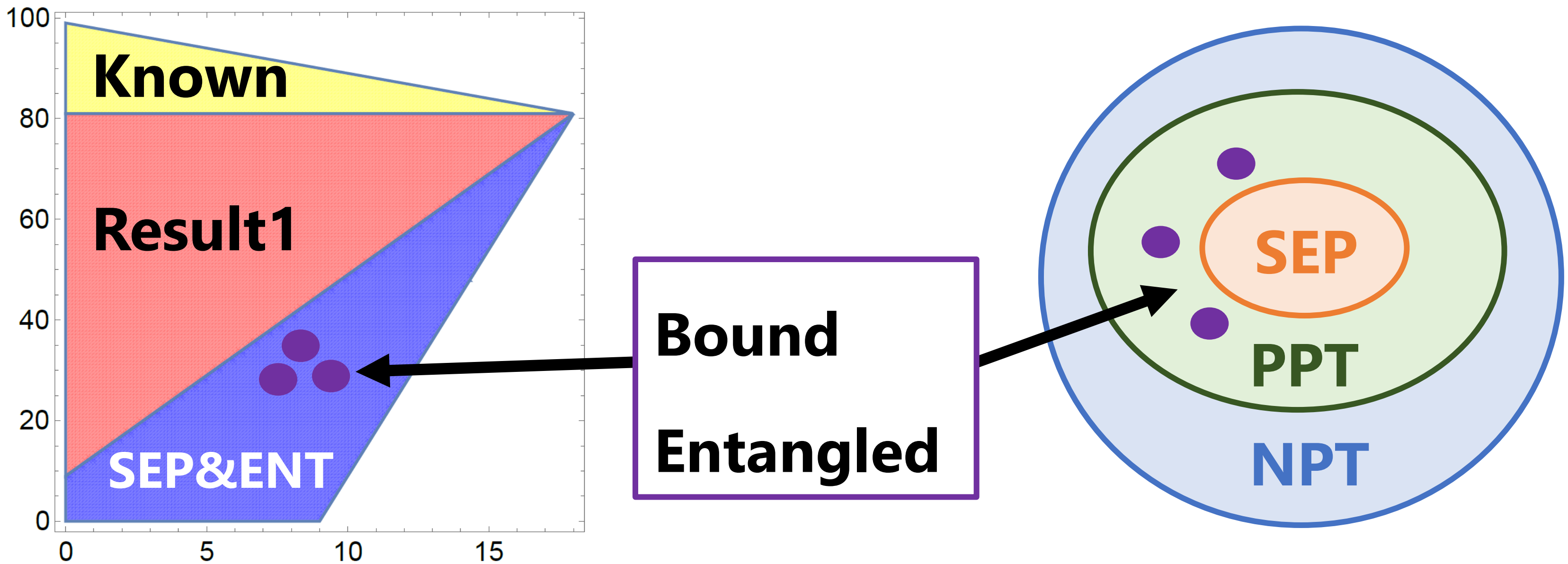
XD Yu, SI, O Gühne, PRL 2021; A Neven, et al npj QI 2021

Bound entanglement

- Operationally it is not distillable: $\varrho_{AB} \otimes \cdots \otimes \varrho_{AB} \not\Rightarrow |\mathrm{Bell}\rangle$
  - very weak form of entanglement in high dimensions

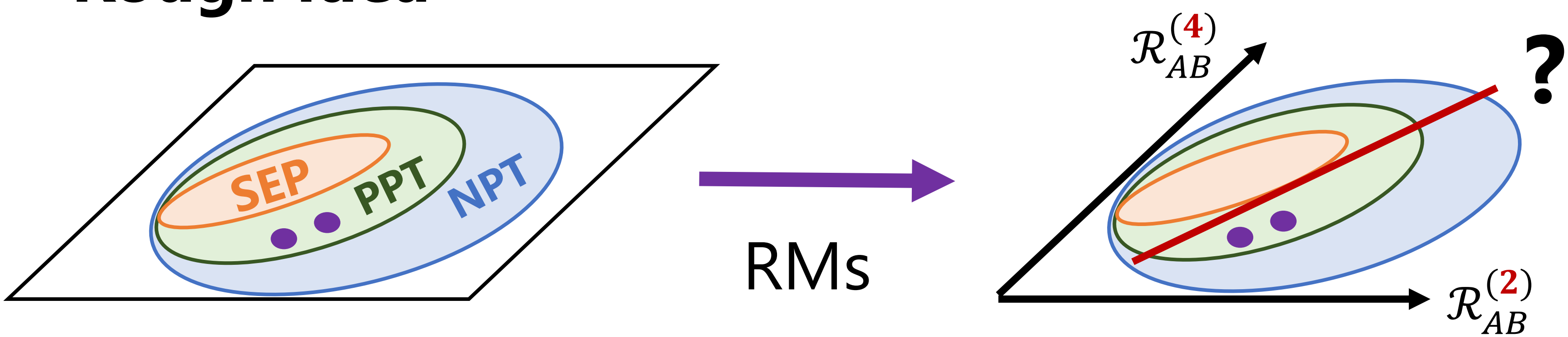
Bound entanglement

- Operationally it is not distillable: $\varrho_{AB} \otimes \cdots \otimes \varrho_{AB} \nRightarrow |\mathrm{Bell}\rangle$
  - very weak form of entanglement in high dimensions
- Mathematically it does not violate the PPT criterion
  - PPT criterion is *strictly* stronger than $\mathrm{tr}[\varrho_{AB}^2] \leq \mathrm{tr}[\varrho_A^2]$



Result 4: BE detection

■ Rough idea



"positive"-separability

"pseudo"-separability
de Vicente JPA 2008

Result 4: BE detection

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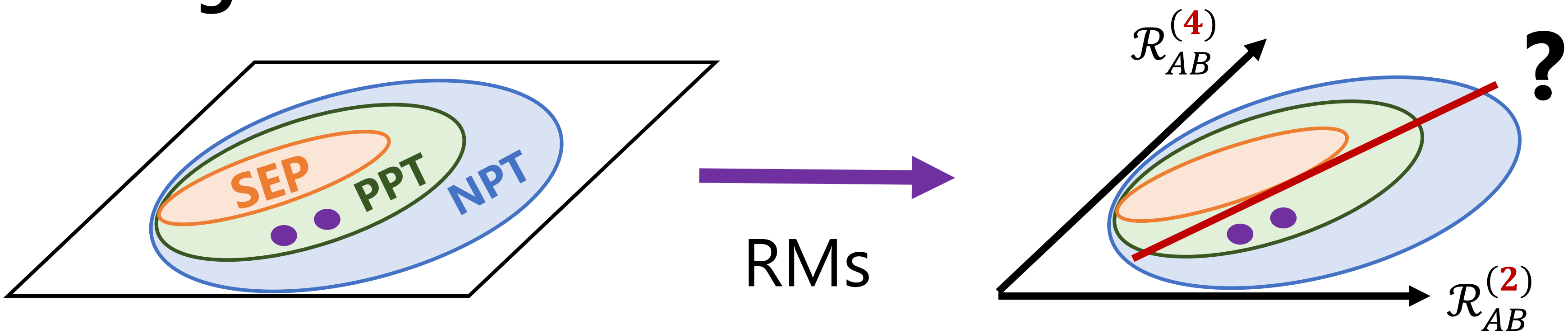

"positive"-separability

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de Vicente JPA 2008

For some measurement observables, it holds that

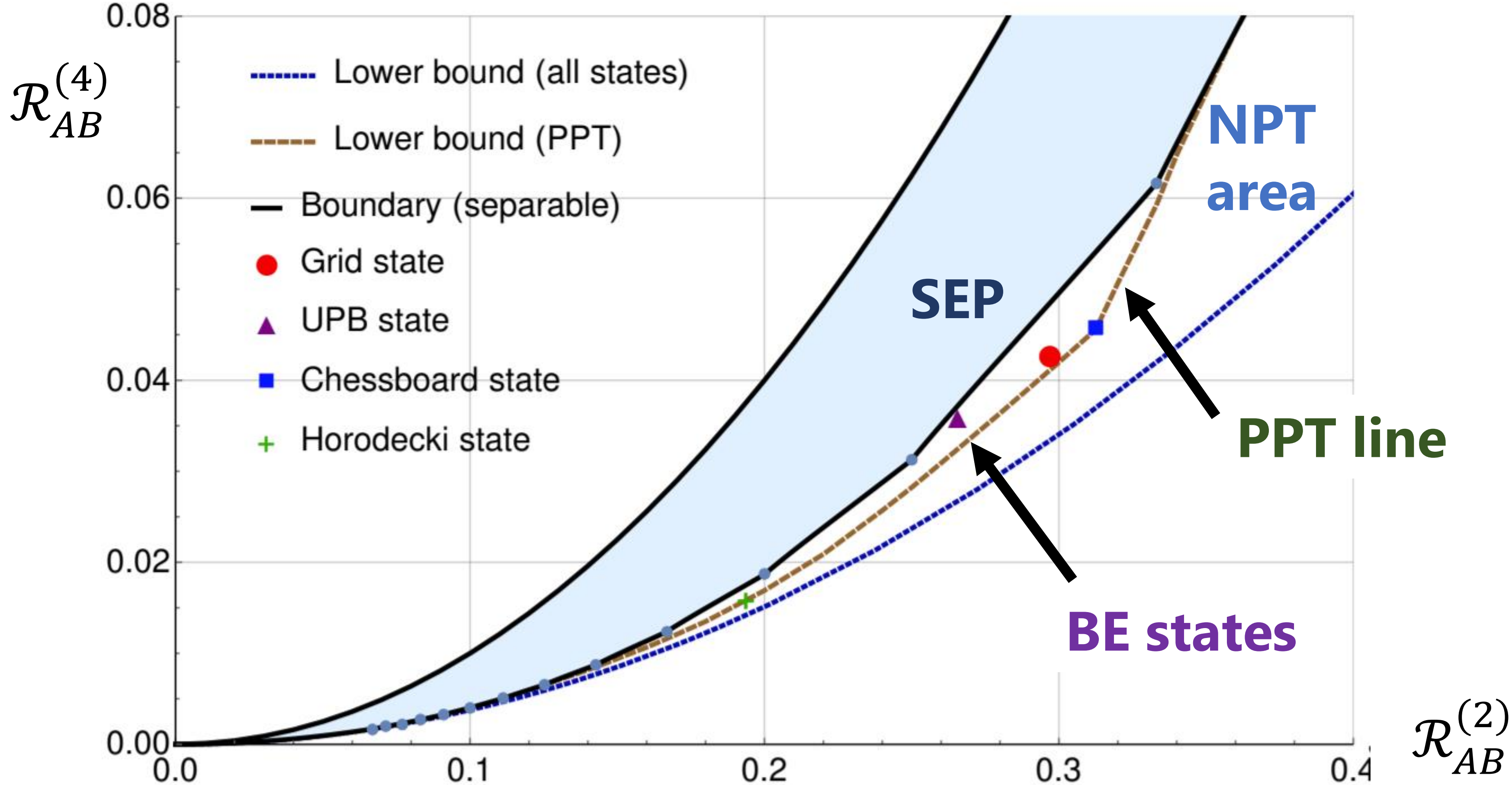
vector of Gell-Mann matrices

$$\mathcal{R}_{AB}^{(r)} \overset{!}{=} N \int_{\mathcal{V}^{d^2-1}} d\boldsymbol{\alpha}_1 \int_{\mathcal{V}^{d^2-1}} d\boldsymbol{\alpha}_2 \{\mathrm{tr}[\varrho_{AB}(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\lambda}_1) \otimes (\boldsymbol{\alpha}_2 \cdot \boldsymbol{\lambda}_2)]\}^r$$

uniformly chosen from pseudo Bloch sphere $\mathcal{V}^{d^2-1}$

Result 4: BE detection

- **Example: 3⊗3 systems**

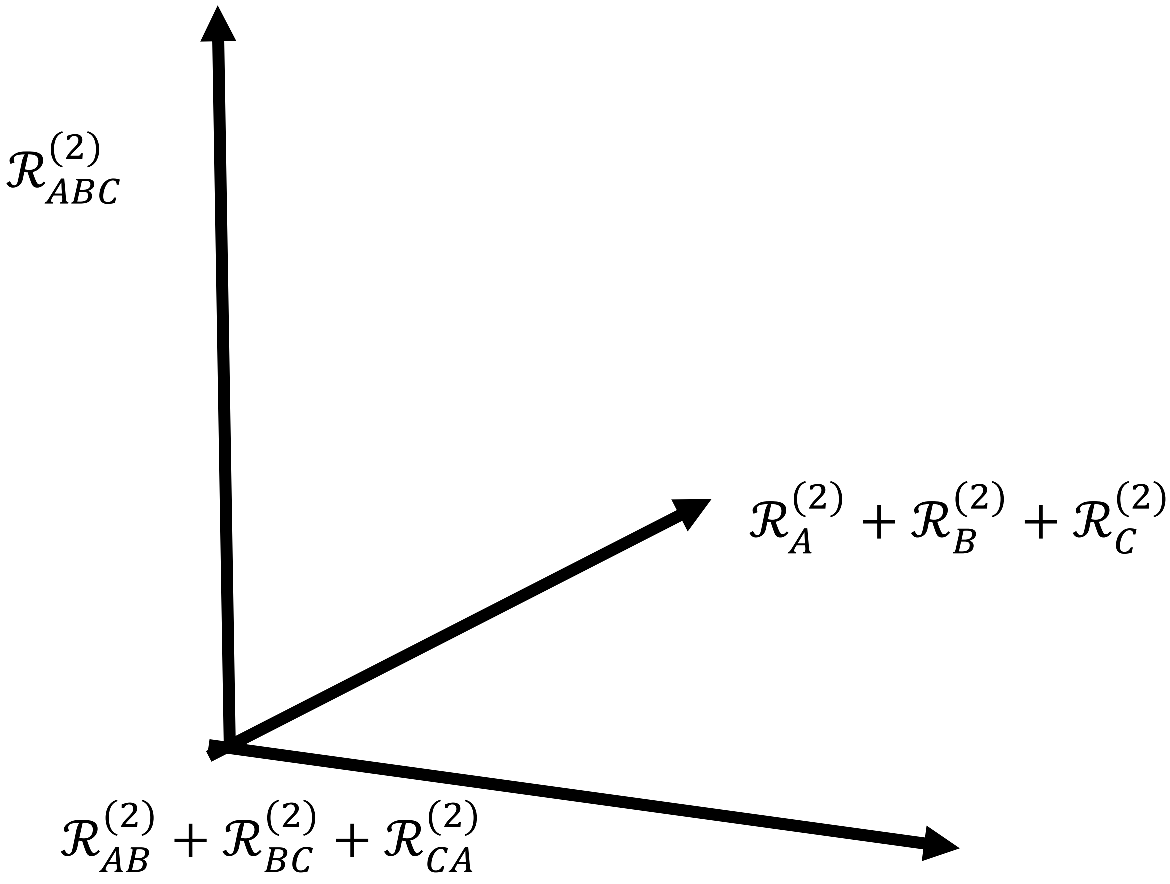


SI, N Wyderka, A Ketterer, O Gühne, PRL 2021

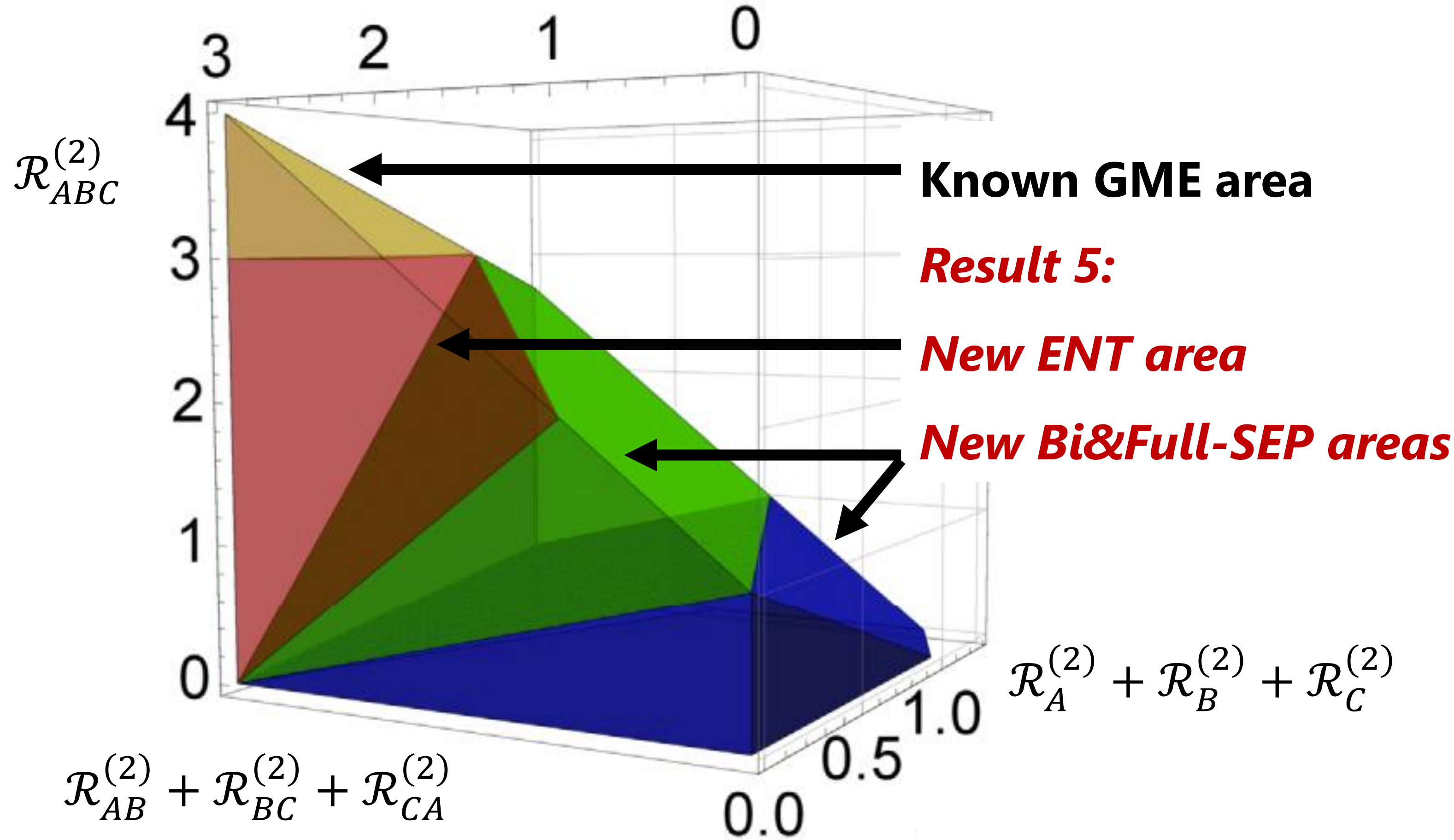
So far so good! What's next?

1. Optimal entanglement detection
2. Schmidt number detection
3. NPT entanglement detection
4. Bound entanglement detection
5. Three-qubit extension?
6. N-qubit extension?
7. Complete characterization?

Result 5: Three-qubit extension



Result 5: Three-qubit extension

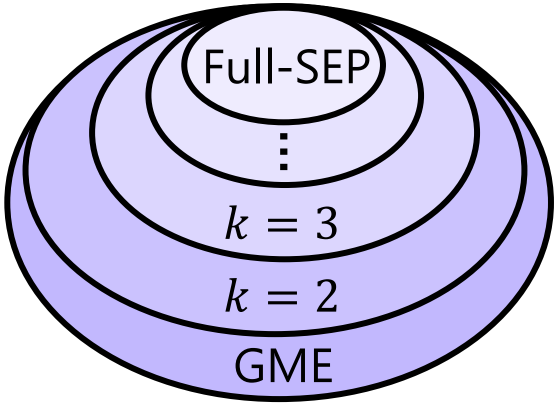


N Wyderka and O Gühne, JPA 2020;
SI, N Wyderka, A Ketterer, O Gühne, PRL 2021

Result 6: N-qubit extension

- Consider number of separable partitions

$$\left|\psi_{k-\mathrm{sep}}\right\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_k\rangle$$



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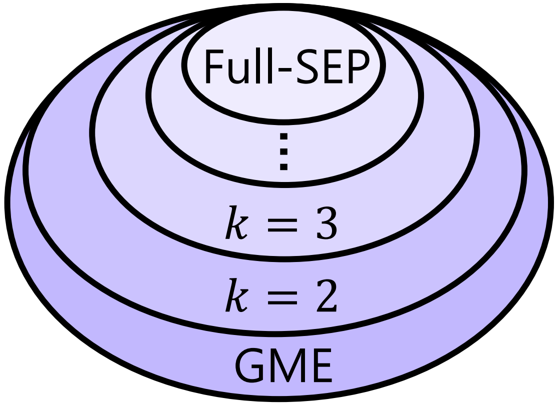
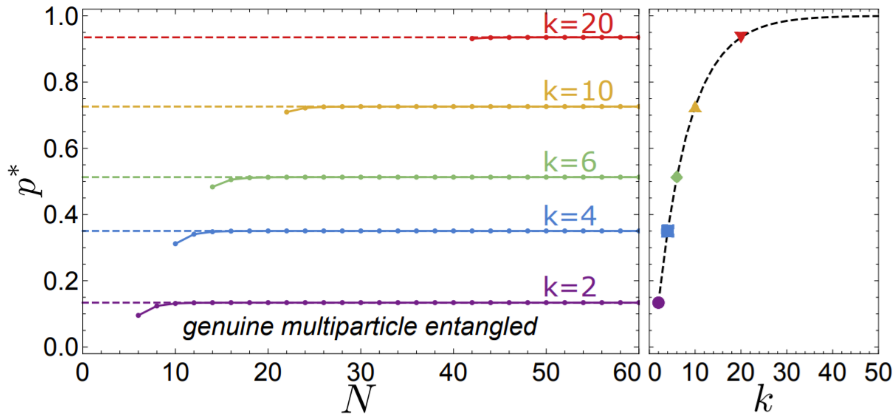


- ***Result 6: Detection of k-separability***

$$\mathcal{R}^{(2)}_{A_1 A_2 \ldots A_N} > f(N, k+1) \Longrightarrow \varrho \text{ is } k\text{-separable}$$

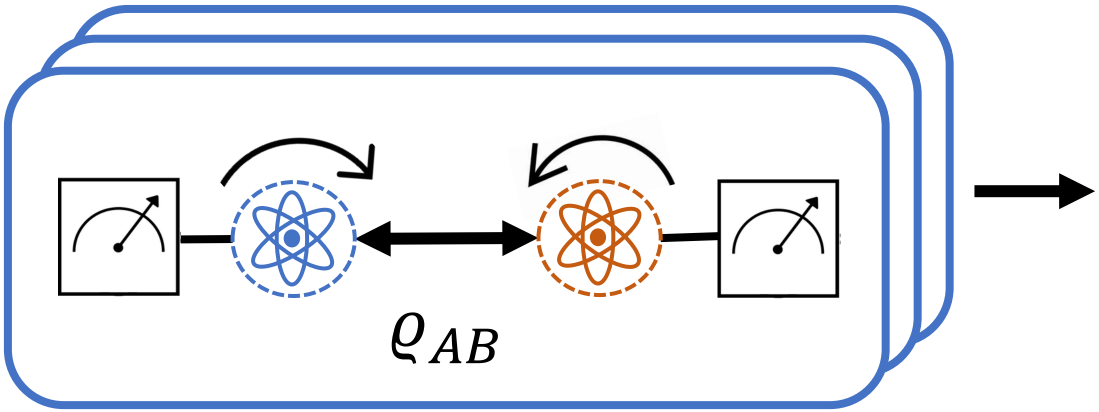


$$\varrho(p) = \frac{p}{2^n} I + (1-p)|\mathrm{GHZ}_N\rangle\langle \mathrm{GHZ}_N|$$

A Ketterer, SI, N Wyderka, O Gühne, PRA (L), 2022

Result 7: Complete characterization

Randomized Measurements



So far

$\alpha^2, \beta^2, \mathrm{tr}(TT^{\mathrm{T}})$

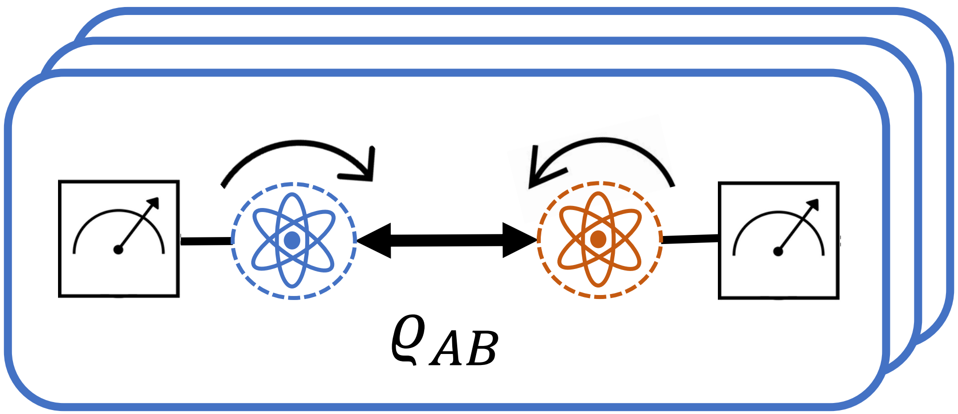
Entanglement detection

Bloch representation of two-qubit state

$$\varrho_{AB} = \frac{1}{4}\left(I^{\otimes 2} + \sum \vec{\alpha} \cdot \vec{\sigma} \otimes I + \sum I \otimes \vec{\beta} \cdot \vec{\sigma} + \sum T_{ij}\sigma_i \otimes \sigma_j\right)$$

Result 7: Complete characterization

Randomized Measurements



!

Result 7

$\alpha^2, \beta^2, \mathrm{tr}(TT^{\mathrm{T}})$

$\vec{\alpha}T\vec{\beta}^T, \det(T)$

$[\vec{\alpha}T]^2, [\vec{\beta}T]^2, \mathrm{tr}(TT^{\mathrm{T}}TT^{\mathrm{T}})$,

and so on...

teleportation fidelity,
CHSH ineq. & PPT criterion

coming soon...

Bloch representation of two-qubit state

$$\varrho_{AB} = \frac{1}{4}\left(I^{\otimes 2} + \sum \vec{\alpha}\cdot\vec{\sigma}\otimes I + \sum I\otimes\vec{\beta}\cdot\vec{\sigma} + \sum T_{ij}\sigma_i\otimes\sigma_j\right)$$

Summary

1. Optimal entanglement detection
2. Schmidt number detection
3. NPT entanglement detection
4. Bound entanglement detection
5. Three-qubit extension
6. N-qubit extension
7. Complete characterization

In preparation…

1. Multiparticle BE detection?
2. Only marginals?
3. Spin squeezing?
4. Quantum metrology?

Other my works

1. ENT change under classicalization
2. Work fluctuations and ENT
3. Q. speed limit for perturbation

Any comments/suggestions are welcome!

Details of BE detection

1. $\varrho_{AB} = \frac{1}{d^2}\left(I^{\otimes 2} + \sum \vec{a}\cdot\vec{\lambda} \otimes I + \sum I \otimes \vec{b}\cdot\vec{\lambda} + \sum T_{ij}\lambda_i \otimes \lambda_j\right)$

2. $\mathcal{R}_{AB}^{(4)}$ & $\mathcal{R}_{AB}^{(2)}$ are (4th and 2nd order) polynomial functions of T_{ij}

3. $\mathcal{R}_{AB}^{(4)}$ & $\mathcal{R}_{AB}^{(2)}$ are invariant under orthogonal rotations O_A, O_B

4. $\varrho_{AB} \in \mathrm{SEP} \rightarrow \|T\|_{\mathrm{tr}} \leq d-1$ and $\|T\|_{\mathrm{tr}} = \left\|O_A T O_B^T\right\|_{\mathrm{tr}} = \sum \tau_i$

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5. max/min $\mathcal{R}_{AB}^{(4)} = 2\sum\tau_i^4 + \left(\mathcal{R}_{AB}^{(2)}\right)^2$

   s.t. $\mathcal{R}_{AB}^{(2)} = \sum\tau_i^2$

   $\sum\tau_i \leq d-1$

6. Get piecewise SEP bounds for any d!