How long does it take to obtain a physical density matrix?

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
Luleå
Oulu
Umeå
White Sea
Arkhangel'sk
Lake Onega
Lake Ladoga
Trondheim
Gulf of Bothnia
Tampere
Helsinki
Turku
Gulf of Finland
St. Petersburg
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
Bergen
Oslo
Stockholm
ALAND ISLANDS
Tallinn
ESTONIA
Gävle
Rockall (U.K.)
HEBRIDES
Stavanger
Skagerrak
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Göteborg
Gotland
LATVIA
Riga
North Atlantic Ocean
North Sea
Aberdeen
Glasgow
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Belfast
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DENMARK
Copenhagen
Malmö
Öland
Baltic Sea
Kaliningrad
RUSSIA
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BELARUS
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SPAIN
Zaragoza
Andorra la Vella
ANDORRA
MONACO
Marseille
SAN MARINO
Florence
Ligurian Sea
Corsica
ITALY
Rome
Adriatic Sea
Skopje
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BALEARIC ISLANDS
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VATICAN CITY
Naples
Tyrrhenian Sea
Tirana
ALB.
Thessaloníki
GREECE
Aegean Sea
Athens
Sevilla
Gibraltar (U.K.)
Málaga
Strait of Gibraltar
Alborán Sea
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Algiers
Mediterranean Sea
Cagliari
Palermo
Sicily
Ionian Sea
Tunis
TUNISIA
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Why tomography is important?

- Many experiments aiming to create many-body entangled states.
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales exponentially with the number of qubits.

Outline

Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions
  T. Monz, P. Schindler, J.T. Barreiro, M. Chwalla, D. Nigg, W.A. Coish, M. Harlander, W. Haensel, M. Hennrich, R. Blatt, arxiv:1009.6126, 2010.

- 10 qubits with photons
  W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, J.-W. Pan, Nature Physics, 6, 331 (2010).

Outline

Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.

$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	...	$A^{(N)}$

- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

Outline

Full quantum state tomography

- The density matrix can be reconstructed from 3^N measurement settings.

Example

For $N = 4$, the measurements are

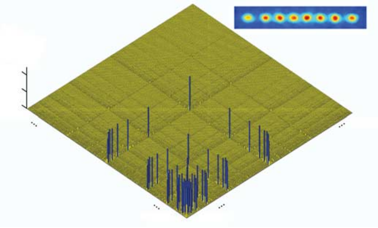
1.	X	X	X	X
2.	X	X	X	Y
3.	X	X	X	Z
		...		
3^4.	Z	Z	Z	Z

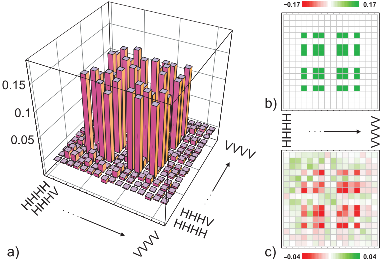
- Note again that the number of measurements scales exponentially in N.

Outline

1. **Motivation**
   - Why quantum tomography is important?
2. **Quantum experiments with multi-qubit systems**
   - Physical systems
   - Local measurements
3. **Full quantum state tomography**
   - Basic ideas and scaling
   - Experiments
   - Approaches to solve the scalability problem
4. **How to obtain a density matrix**
5. **Extra slides**

Experiments with ions and photons





- 8 ions: H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).
- 4 photons: N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. 98, 063604 (2007).
- 6 photons: C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Phys. Rev. Lett. 113, 040503 (2014).

Outline

Approaches to solve the scalability problem

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.
  S.T. Flammia *et al.,* arxiv:1002.3839; M. Cramer, M.B. Plenio, arxiv:1002.3780; O. Landon-Cardinal *et al.,* arxiv:1002.4632.

- If the state is of low rank, we need fewer measurements.
  D. Gross et al., Phys. Rev. Lett. 105, 150401 (2010).

- We make tomography in a subspace of the density matrices (our approach).
  G. Tóth *et al.* , Phys. Rev. Lett. 105, 250403 (2010); T. Moroder *et al.*, New J. Phys. 14, 105001 (2012); C. Schwemmer *et al.*, Phys. Rev. Lett. 113, 040503 (2014)

Obtain a density matrix

- The density matrix can be decomposed into correlations as

$$\varrho = \frac{1}{2^n} \sum_{\boldsymbol{\mu}} T_{\boldsymbol{\mu}} \sigma_{\boldsymbol{\mu}},$$

  where $\sigma_{\boldsymbol{\mu}} = \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \cdots \otimes \sigma_{\mu_n}$, $\mu_i \in \{0, 1, 2, 3\}$, and σ_0 denotes the identity matrix.

- The correlation matrix is defined as $T_{\boldsymbol{\mu}} = \langle \sigma_{\boldsymbol{\mu}} \rangle$.

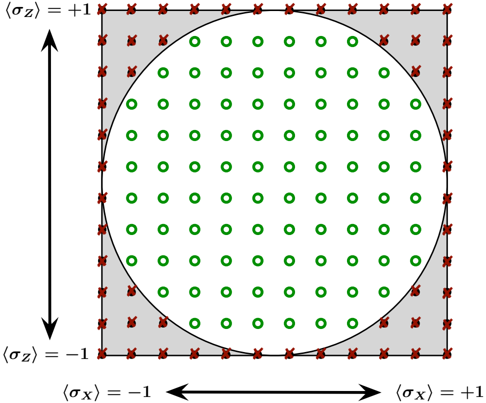
- How can we obtain the estimate $\tilde{\varrho}$? We just measure $T_{\boldsymbol{\mu}}$.

Obtain a density matrix II

- How can we obtain the estimate $\tilde{\varrho}$? We just measure $T_{\boldsymbol{\mu}} = \langle \sigma_{\boldsymbol{\mu}} \rangle$.
- Problem: we have finite number of measurements.

Obtain a density matrix III

- 1 qubit, 11 measurements.



[R. Blume-Kohout, arXiv:quant-ph/0611080]

Why negative eigenvalues are a problem?

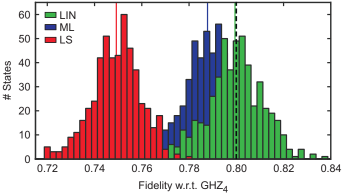
- We cannot calculate fidelities with a mixed state, entropies, purity, entanglement, etc.
- We can still calculate the fidelity with a pure state. This is just the expectation value of a projector.

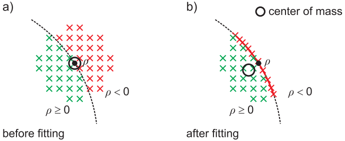
Fitting

- Method to get rid of the negative eigenvalues of ϱ.
- Find the physical density matrix in a best agreement with the experimental data.
- Main methods: maximum likelihood, least squares.

Problems with fitting

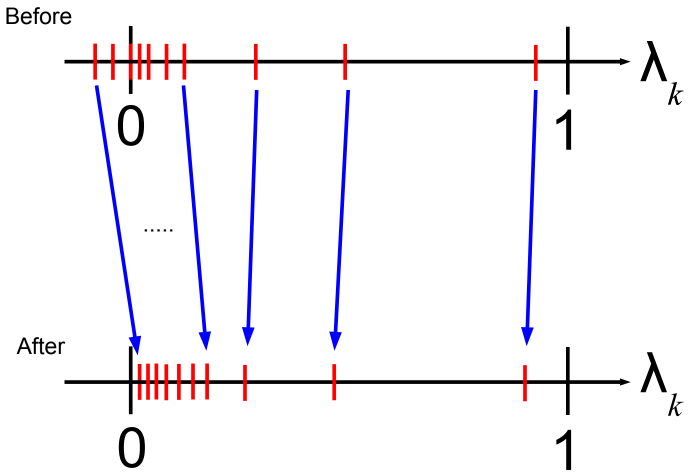
- Fidelity changes, bias, detection of fake entanglement





[Schwemmer et al., PRL 114, 080403 (2015).]

Problems with fitting



Small eigenvalues increase Large eigenvalues decrease

Let us analyze the problem

- Completely mixed state

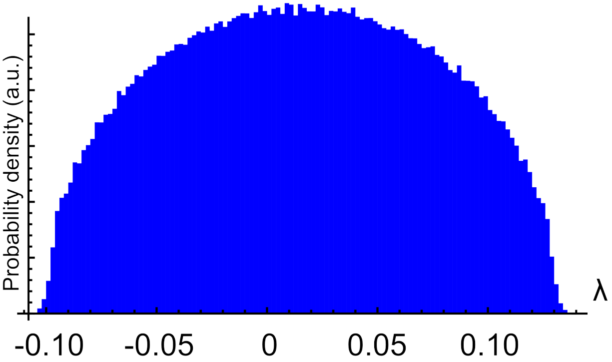
$$\varrho_{\rm wn} = \frac{1}{2^n}\sigma_{0,0,...,0} = \frac{1}{2^n}\mathbb{1}$$

  with 2^n degenerate eigenvalues $\lambda_i = 1/2^n$.

- We use overcomplete tomography, which is based on measuring the Pauli correlations.

Distribution of eigenvalues

- Consider n = 6 qubit maximally mixed state
- Simulate N = 100 measurements per setting
- Estimate density matrix
- Repeat 10 000 times
- Histogram of eigenvalues



Distribution of eigenvalues

- Consider n = 6 qubit maximally mixed state
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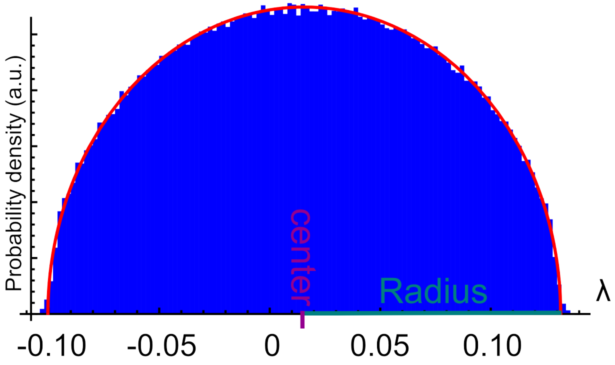


How long do we have to measure to get a physical state?

- Pure state mixed with white noise

$$\varrho_q = q|\psi\rangle\langle\psi| + (1-q)\varrho_{\text{cm}}.$$

- The center is shifted to

$$c_q = \frac{1-q}{2^n - r}.$$

- The radius is

$$R = 2\sqrt{\frac{10^n - 1}{12^n}}\frac{1}{\sqrt{N}} \approx 2\left(\frac{5}{6}\right)^{\frac{n}{2}}\frac{1}{\sqrt{N}}.$$

- Physical ϱ if

$$R \leq c_q \Rightarrow N \geq N_0 = 4\left(\frac{5}{6}\right)^n \left(\frac{2^n - 1}{1-q}\right)^2.$$

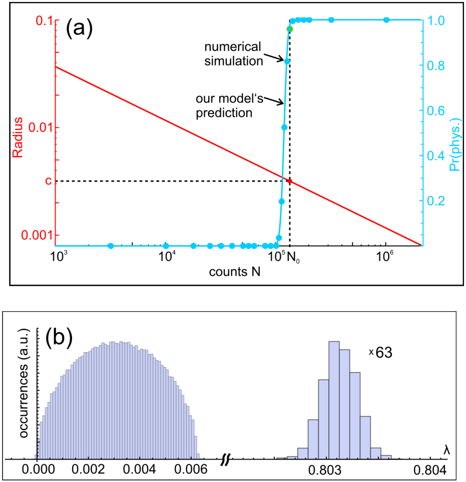
How long do we have to measure to get a physical state? II

- The minimum number of measurements needed is

$$N_0 = 4\left(\frac{5}{6}\right)^n \left(\frac{2^n - 1}{1 - q}\right)^2 .$$

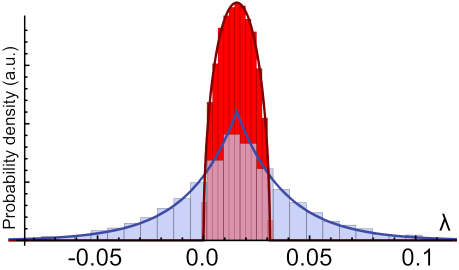
How long do we have to measure to get a physical state? III

- Six-qubit GHZ state mixed with $q = 0.2$ white noise



Other type of tomography

- Not all tomographies lead to a Wigner semicircle

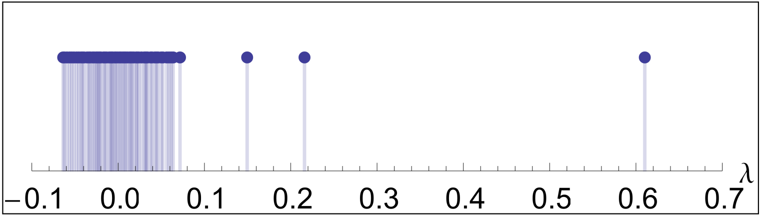


Hypothesis testing

- We prepare a six-qubit Dicke state

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + ... + |111000\rangle).$$

- Quantum state tomography with around 230 events per setting.
- Hypothesis: 3 eigenvalues + noise. Is this correct?

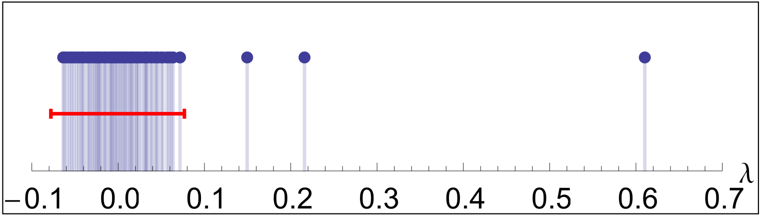


Hypothesis testing II

- We prepare a six-qubit Dicke state

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + ... + |111000\rangle).$$

- Quantum state tomography with around 230 events per setting.
- Hypothesis: 3 eigenvalues + noise. Is this correct?

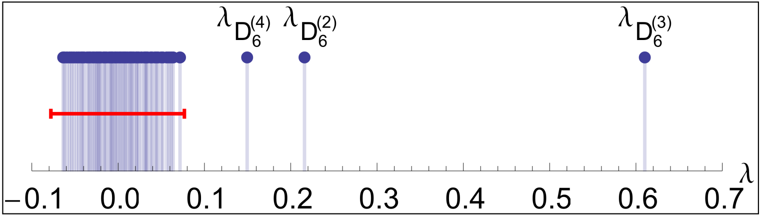


Hypothesis testing III

- We prepare a six-qubit Dicke state

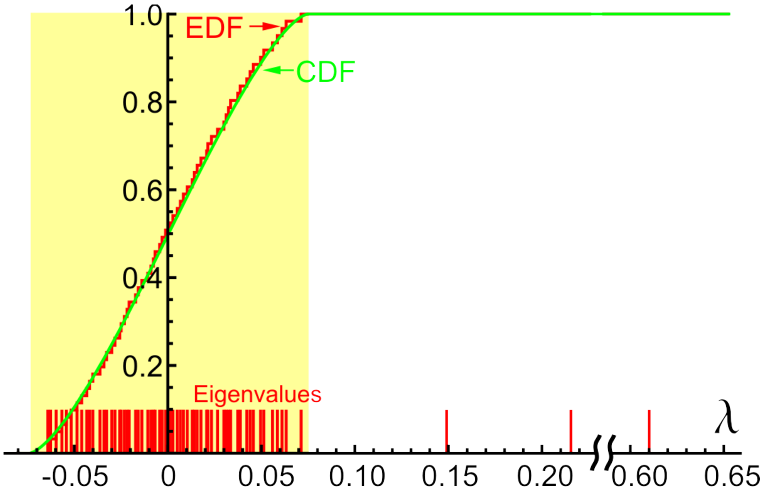
$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + ... + |111000\rangle).$$

- Quantum state tomography with around 230 events per setting.
- Hypothesis: 3 eigenvalues + noise. Is this correct?

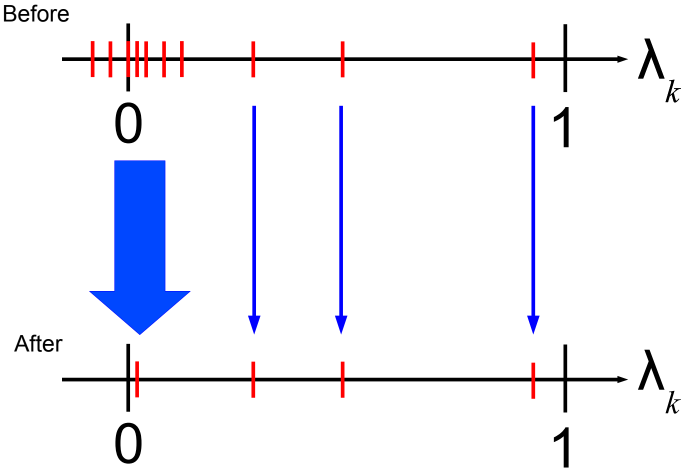


Is the hypothesis correct?

- Empirical distribution function (EDF) vs. Cumulative distribution function (CDF) of the Wigner semicircle



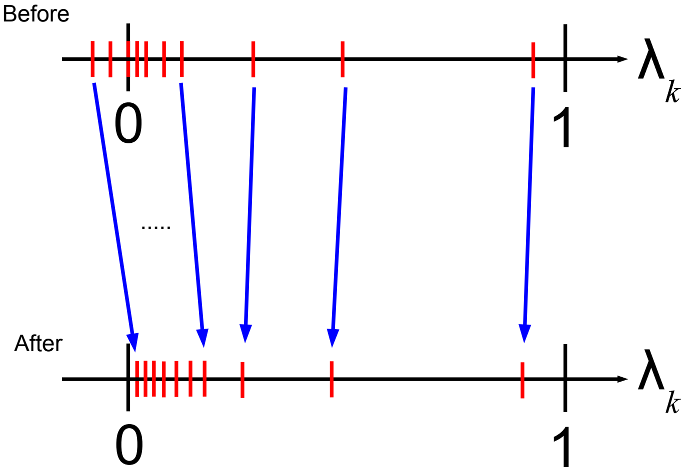
Our method



Small eigenvalues are replaced by their average

Large eigenvalues do not change

Just to compare: old method



Small eigenvalues increase Large eigenvalues decrease

Summary

- We discussed the distribution of the eigenvalues of density matrices obtained from tomography.
- We suggested a method to get rid of negative eigenvalues.
- I thank Lukas Knips for most of the figures for this talk.

See:
L. Knips, C. Schwemmer, N. Klein, J. Reuter, G. Tóth, and H. Weinfurter,

How long does it take to obtain a physical density matrix?, arxiv:1512.06866.

THANK YOU FOR YOUR ATTENTION!

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EUSKO JAURLARITZA
GOBIERNO VASCO

Derivation (slide from Lukas Knips)

- Concept: compare moments of eigenvalue distribution to moments of ideal semicircle function

- Define semicircle distribution

$$f_{c,R}(x) = \underbrace{\frac{2}{\pi R^2}}_{Z} \sqrt{(x-c)^2 - R^2}$$

with (even) moments

$$m_2^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^2 x = \left(\frac{R}{2}\right)^2,$$
$$m_4^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^4 x = 2\left(\frac{R}{2}\right)^4,$$
$$m_6^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^6 x = 5\left(\frac{R}{2}\right)^6,$$
$$m_8^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^8 x = 14\left(\frac{R}{2}\right)^8.$$

Using the *Catalan* numbers

$$\mathcal{C}_{j+1} = \mathcal{C}_j \frac{2(2j+1)}{j+2}$$

we obtain

$$m_{2k}^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^{2k} x = \mathcal{C}_k \left(\frac{R}{2}\right)^{2k}$$

- Odd (centralized) moments vanish

- Goal: reproduce Catalan numbers in distribution of eigenvalues

- Calculate all moments of eigenvalue distribution:

$$\begin{aligned} m_k^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n} \mathbb{E}\left[\lambda_i^k\right] \\ &= \frac{1}{2^n}\mathbb{E}\left[\sum_{i=1}^{2^n}\lambda_i^k\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\mathrm{Tr}\left(D^k\right)\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\mathrm{Tr}\left(\left(U^\dagger \varrho U\right)^k\right)\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\mathrm{Tr}\left(\varrho^k\right)\right] \end{aligned}$$

- Second moment of (centered) distribution:

$$\begin{aligned} m_2^{\mathrm{ev}} &= \frac{1}{2^{3n}}\sum_{\vec\mu,\vec\nu}\mathbb{E}\left[T_{\vec\mu}T_{\vec\nu}\right]2^n\delta_{\vec\mu,\vec\nu} \\ &= \frac{2^n}{2^{3n}}\sum_{\vec\mu}\mathbb{E}\left[T_{\vec\mu}^2\right] \end{aligned}$$

overcomplete Pauli scheme:

$$\begin{aligned} m_2^{\mathrm{ev}} &= \frac{1}{4^n N}\sum_{j=0}^{n-1}\binom{n}{j}\frac{3^{n-j}}{3^j} \\ &= \frac{10^n-1}{12^n}\frac{1}{N}. \end{aligned}$$

with n qubits, N events per basis element.

- Comparision of m_2^{sc}, m_2^{ev} yields:

$$R = 2\sqrt{\frac{10^n-1}{12^n}}\frac{1}{\sqrt{N}}$$

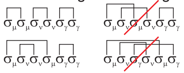
- Fourth moment:

$$\begin{aligned} m_4^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^4\right] \\ &= \frac{1}{2^{5n}}\sum_{\vec\mu,\vec\nu,\vec\gamma,\vec\lambda}\mathbb{E}\left[T_{\vec\mu}T_{\vec\nu}T_{\vec\gamma}T_{\vec\lambda}\right] \\ &\qquad \cdot \mathrm{Tr}\left(\sigma_{\vec\mu}\sigma_{\vec\nu}\sigma_{\vec\gamma}\sigma_{\vec\lambda}\right) \\ &= \frac{1}{2^{5n}}\frac{1}{2!}\sum_{\vec\mu}\sum_{\vec\nu:\{\vec\nu\neq\vec\mu\}}\mathbb{E}\left[T_{\vec\mu}^2T_{\vec\nu}^2\right] \\ &\qquad \cdot \mathrm{Tr}\left(\sum_{i=1}^{6}\mathcal{P}_i\left(\sigma_{\vec\mu}\sigma_{\vec\mu}\sigma_{\vec\nu}\sigma_{\vec\nu}\right)\right) \end{aligned}$$

- Sixth moment:

$$\begin{aligned} m_6^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^6\right] \\ &\approx \frac{1}{2^{7n}}\frac{1}{3!}\sum_{\vec\mu}\sum_{\vec\nu:\{\vec\nu\neq\vec\mu\}}\sum_{\vec\gamma:\{\vec\gamma\neq\vec\nu,\vec\gamma\neq\vec\mu\}} \\ &\qquad \mathbb{E}\left[T_{\vec\nu}^2T_{\vec\mu}^2T_{\vec\gamma}^2\right]\cdot \\ &\qquad \mathrm{Tr}\left(\sum_{i=1}^{90}\mathcal{P}_i\left(\sigma_{\vec\mu}\sigma_{\vec\mu}\sigma_{\vec\nu}\sigma_{\vec\nu}\sigma_{\vec\gamma}\sigma_{\vec\gamma}\right)\right) \end{aligned}$$

non-crossing crossing



- Only non-crossing partitions (amount given by *Catalan* numbers) contribute:

$$m_k^{\mathrm{ev}} = \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^{2k}\right] = \frac{\mathcal{C}_k}{N^k} \quad \square$$