Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles

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Workshop on continuously monitored quantum systems,
Warsaw, 12 December 2023

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Greenland Sea
Denmark Strait
Hammerfest
Tromsø
Murmansk
Norwegian Sea
Kiruna
Reykjavík
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Umeå
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Lake Onega
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SWEDEN
Tórshavn
Faroe Islands (DENMARK)
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SAN MARINO
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Tyrrhenian Sea
Aegean Sea
GREECE
Athens
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Sevilla
Cagliari
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Málaga
Palermo
Mediterranean Sea
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
0 300 Kilometers
0 20 300 Miles

Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Entanglement detection close to singlet states**
   - Singlet obtained from squeezing of J_x, J_y, and J_z
   - Applications for gradiometry
4. **Detecting multipartite entanglement of Dicke states**
   - Dicke state realized with a BEC of two-state atoms
5. **Detecting bipartite entanglement of Dicke states**
   - Bipartite entanglement from multipartite entanglement in BEC
   - Dicke state in a double well
   - Problems to solve in an experiment
   - Entanglement detection in Dicke states

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Why is this challenging?

- It could happen that
  - it is not possible to create large scale entanglement in a system that is not completely isolated.
  - such entanglement is created, but we cannot verify its presence, since we can measure few things.

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Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \ldots \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

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Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \tfrac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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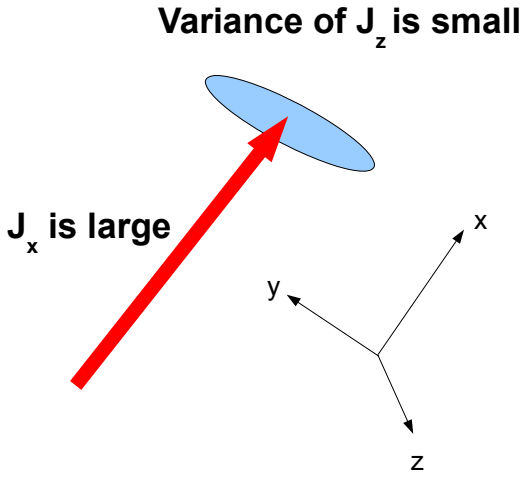
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_S^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

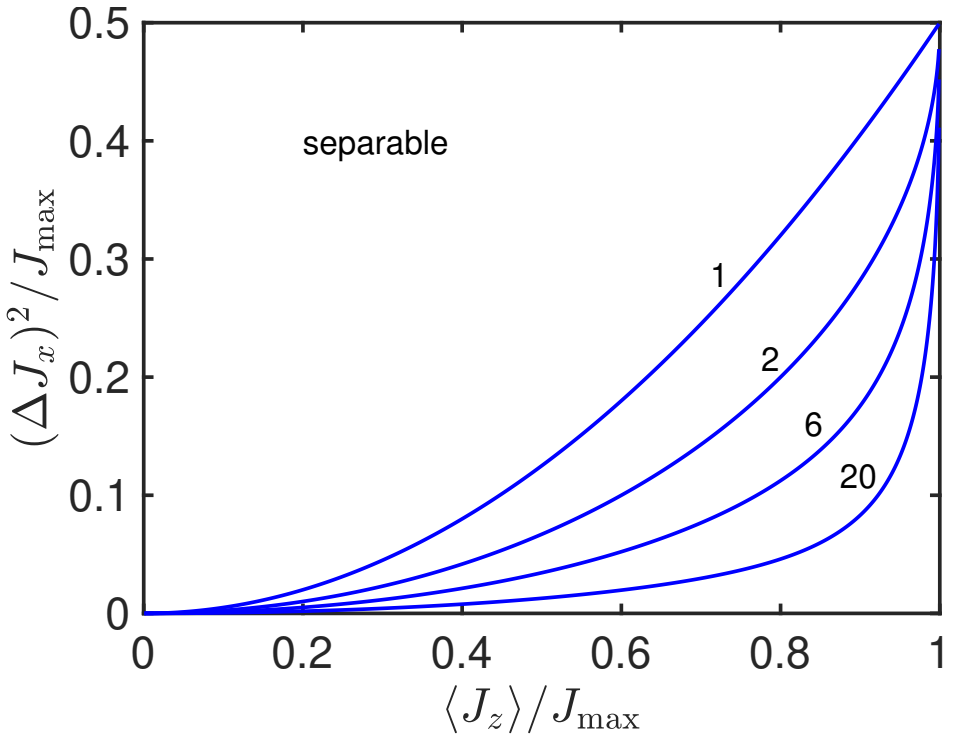
- If $\xi_S^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

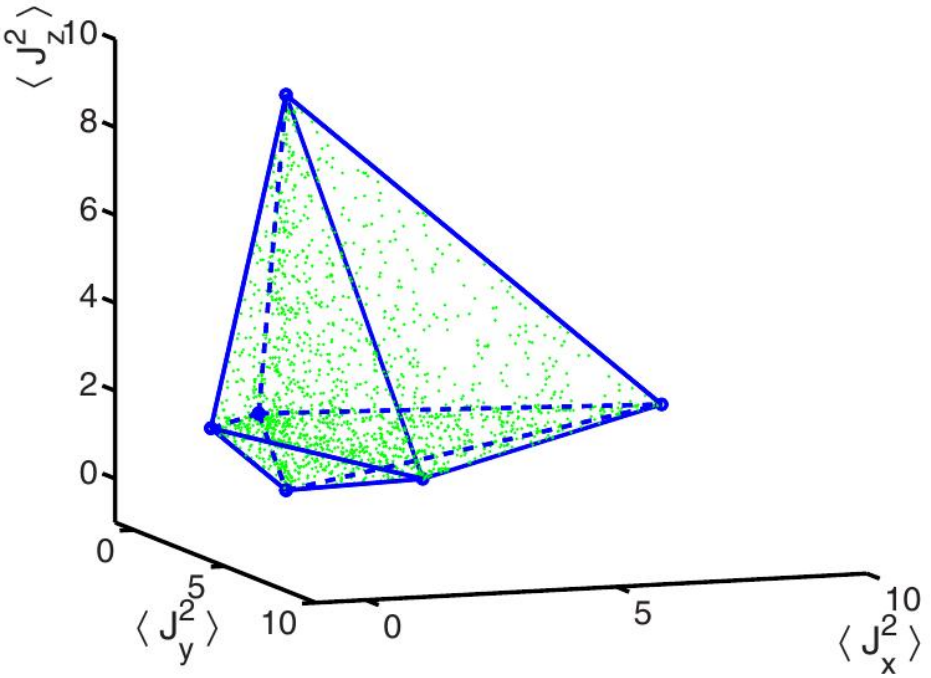
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

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Singlet state

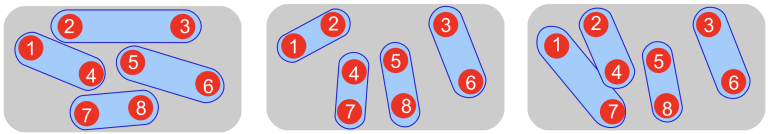
- Singlet states are ground states of antiferromagnetic Hamiltonians.
- The permutationally invariant singlet is

$$\varrho_{\text{singlet}} \propto \lim_{T \to 0} e^{-\frac{J_x^2 + J_y^2 + J_z^2}{T}}.$$

- Such a state has been created in cold atoms.

Singlet state experiments

- Creating singlets by squeezing the uncertainties of spin components → incoherent process, "creating entanglement with decoherence".



G. Tóth and M. W. Mitchell, Generation of macroscopic singlet states in atomic ensembles, New J. Phys. 12 053007 (2010).

- Entanglement in singlet states in $> 10^6$ cold Rb atoms.

N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, GT, R. J. Sewell, and M. W. Mitchell, Generation of Macroscopic Singlet States in a Cold Atomic Ensemble, Phys. Rev. Lett. 113, 093601 (2014).

- Entanglement in singlet states in $> 10^{13}$ hot alkali atoms.

J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, GT, and Morgan W. Mitchell, Measurement-induced, spatially-extended entanglement in a hot, strongly-interacting atomic system,

Nat. Commun. 11, 2415 (2020).

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Applications of singlets

- Singlets are invariant under homogeneous fields.
- Gradient field destroys the singlet → Gradient metrology.

I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and GT, Precision bounds for gradient magnetometry with atomic ensembles, Phys. Rev. A 97, 053603 (2018).

S. Altenburg, M. Oszmaniec, S. Wölk, O. Gühne, Estimation of gradients in quantum metrology, Phys. Rev. A 96, 042319 (2017).

I. Urizar-Lanz, P. Hyllus, I. L. Egusquiza, M. W. Mitchell, and GT, Macroscopic singlet states for gradient magnetometry, Phys. Rev. A 88, 013626 (2013).

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spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Summing over all permutations. $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al.*, Nat. Phys. 2012.; L. Xin *et al.*, Phys. Rev. Lett 2023; Y-Q. Zhou, *et al.*, PNAS 2018.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for "50 year of Bell's theorem", 2014.

- ... have high levels of bipartite entanglement

$$E_F \approx \log_2(N)/2 \equiv \log_2(\sqrt{N}).$$

Note that for a maximally entangled state,

$$E_F = \log_2(d).$$

J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi,
Phys. Rev. A 67, 022112 (2003).

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

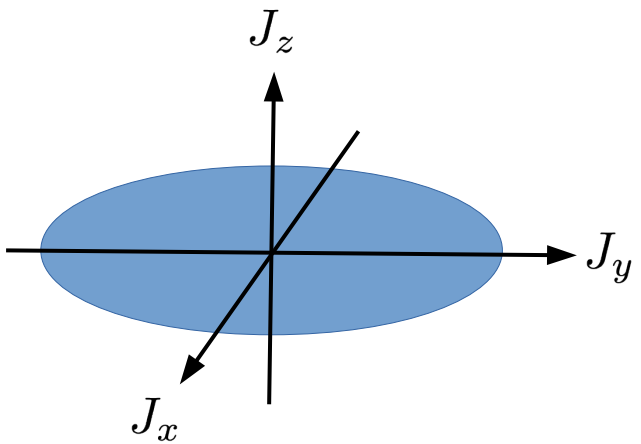
holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$

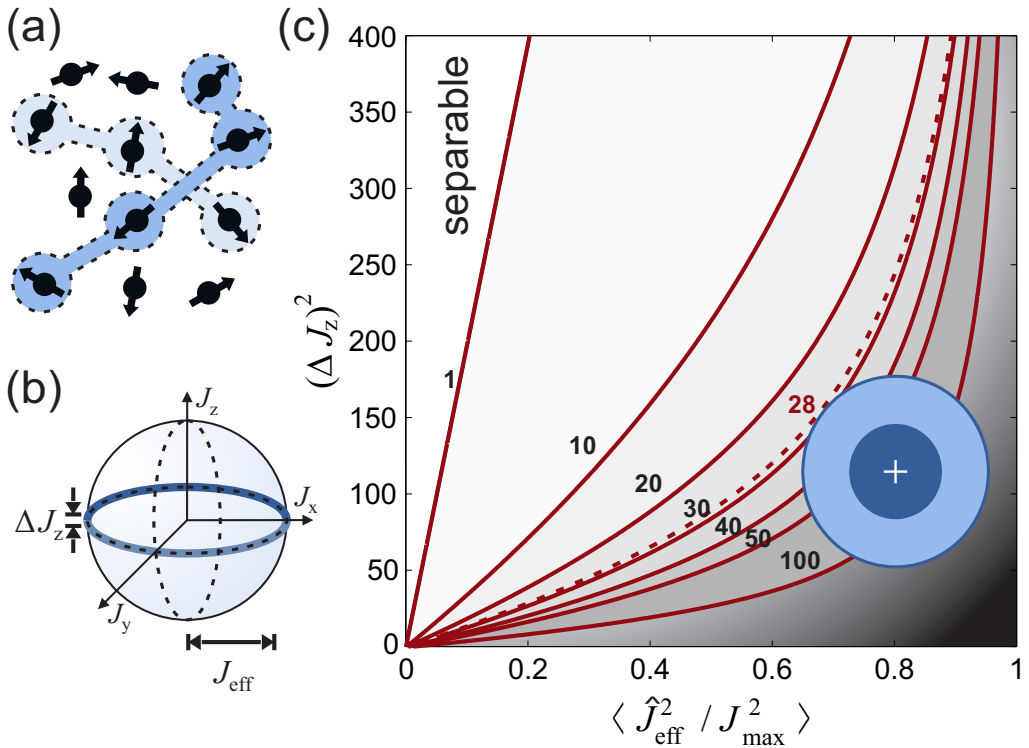
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\rm eff} = J^2_x + J^2_y, \quad J_{\rm max} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014); Vitagliano *et al.*, NJP 2017.

Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can even detect multipartite entanglement knowing only two-body correlations!

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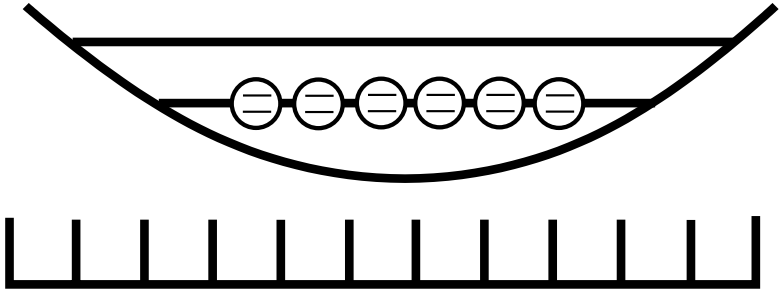
Bipartite entanglement from bosonic multipartite entanglement

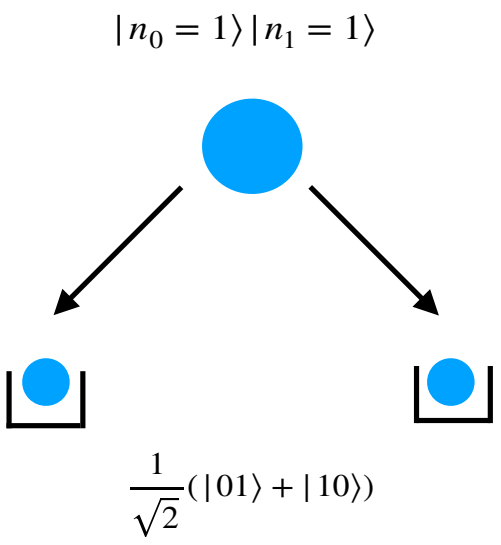
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument

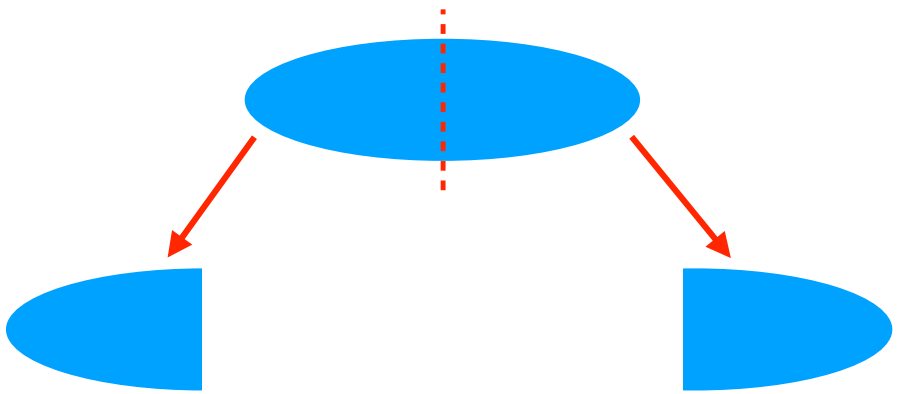


See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.
- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

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Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in the spin state $|j_z = 0\rangle$.
- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

  Tunneling from mode 0 to the mode $+1$ and -1.

- Understanding the tunneling process

$$|j_z = 0\rangle|j_z = 0\rangle \rightarrow \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$$= \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

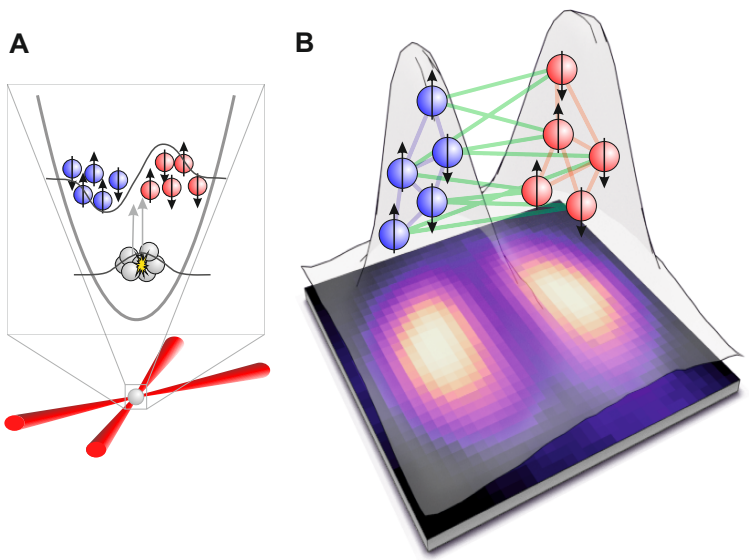
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Correlations for Dicke states

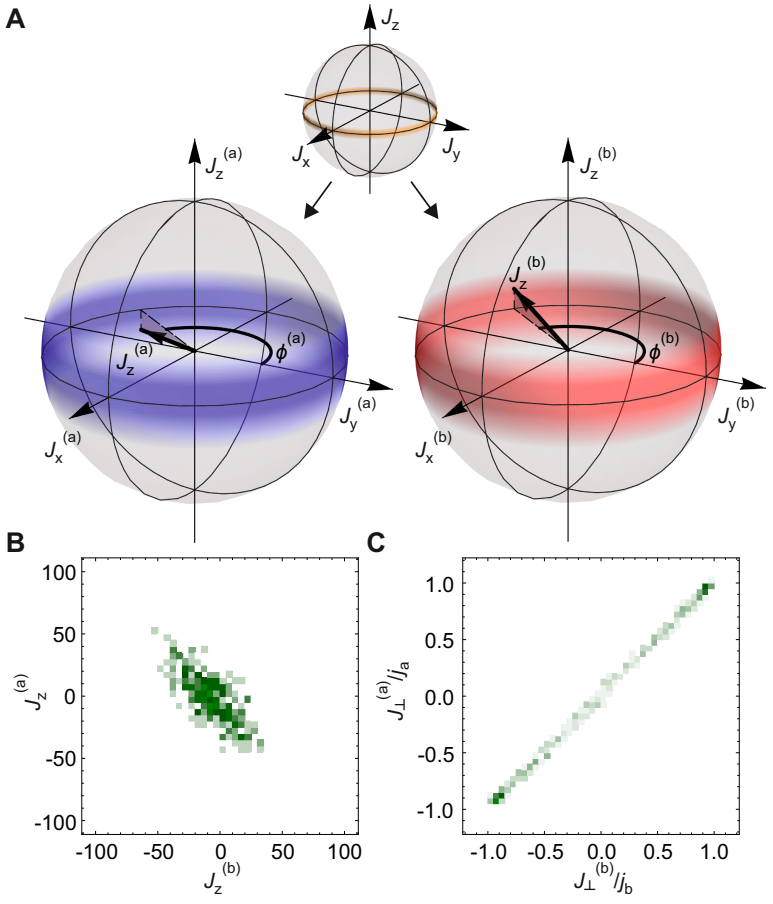
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_{\mathrm{x}}^{a} - J_{\mathrm{x}}^{b}))^2 &\approx 0, \\
(\Delta(J_{\mathrm{y}}^{a} - J_{\mathrm{y}}^{b}))^2 &\approx 0, \\
(\Delta J_{\mathrm{z}})^2 &= 0.
\end{aligned}
$$

- Measurement results on well "b" can be predicted from measurements on "a"

$$
\begin{aligned}
J_{\mathrm{x}}^{b} &\approx J_{\mathrm{x}}^{a}, \\
J_{\mathrm{y}}^{b} &\approx J_{\mathrm{y}}^{a}, \\
J_{\mathrm{z}}^{b} &= -J_{\mathrm{z}}^{a}.
\end{aligned}
$$

Correlations for Dicke states - experimental results



Here, $J_{\perp}^{(n)} = \cos\alpha J_{x}^{(n)} + \sin\alpha J_{y}^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

Outline

1. Motivation
   - Why entanglement is important?
2. Spin squeezing and entanglement
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. Entanglement detection close to singlet states
   - Singlet obtained from squeezing of J_x, J_y, and J_z
   - Applications for gradiometry
4. Detecting multipartite entanglement of Dicke states
   - Dicke state realized with a BEC of two-state atoms
5. Detecting bipartite entanglement of Dicke states
   - Bipartite entanglement from multipartite entanglement in BEC
   - Dicke state in a double well
   - Problems to solve in an experiment
   - Entanglement detection in Dicke states

Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Consider a density matrix

$$\varrho = \sum_{j_a, j_b} Q_{j_a, j_b} \varrho_{j_a, j_b},$$

where ϱ_{j_a, j_b} are states with

$$N_a = 2j_a, \quad N_b = 2j_b$$

particles in the two wells, Q_{j_a, j_b} are probabilities.

- ϱ is entangled iff at least one of the ϱ_{j_a, j_b} is entangled.

Problem 1: Varying particle number II

- Even if we have a constant total particle number, the ensemble will not be evenly split.
- Probability distribution for having $N/2 + x$ particles

$$p_x = 2^{-N}\binom{N}{N/2+x}.$$

- Variance

$$\mathrm{var}(N_\mathrm{a}) = \mathrm{var}(x) = \langle x^2 \rangle = \frac{N}{4.}$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left(\frac{N}{8} + \frac{1}{2}x^2\right) = \frac{N}{8} + \frac{1}{2}\mathrm{var}(x) = \frac{N}{4}, \quad l = x, y.$$

Twice as large due to that the ensemble is not evenly split.

Problem 1: Varying particle number III

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}$$

  for $l = x, y$.

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

Problem 1: Varying particle number IV

- Let use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a+1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b+1)}} J_l^b \approx 2\left(\frac{J_l^a}{N_a} - \frac{J_l^b}{N_b}\right)$$

  for $l = x, y$.

- For the variance of $\mathcal{J}_l$ we obtain for $N/2 + x : N/2 - x$ splitting

$$(\Delta\mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

  After splitting $|x| \lesssim \sqrt{N/4}$.

- We have

$$(\Delta\mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

  $(\Delta\mathcal{J}_l^-)^2$ is not sensitive to the fluctuation of x if N is large.

Problem 2: States are not always symmertic in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must hadle this.

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Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2 + \left\langle J_y\right\rangle^2).$$

  Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle + \left\langle J_y^2\right\rangle).$$

- Simple algebra yields

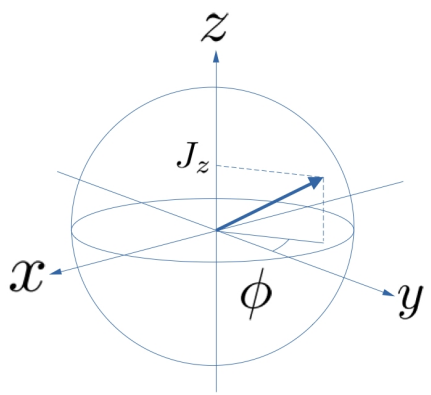
$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_z)^2+\frac{1}{4}\right]}_{\sim \text{ fluctuation of } J_z}\times\underbrace{\frac{(\Delta J_x)^2+(\Delta J_y)^2}{\langle J_x^2\rangle+\langle J_y^2\rangle}}_{\sim \text{ phase fluctuation}}\geq\frac{1}{4}.$$



Handwaving description:
J_z and ϕ cannot be defined both with high accuracy.

Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_m^n = \frac{J_m^n}{\sqrt{j_n(j_n+1)}} \approx \frac{J_m^n}{N_n},$$

where $m = x, y$ and $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables $\rightarrow$ resistance to experimental imperfections.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x)^2 + (\Delta \mathcal{J}_y)^2\right] \geq \frac{1}{4}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle .$$

The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result I

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{I}_x^-)^2 + (\Delta \mathcal{I}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{I}_x^2 + \mathcal{I}_y^2 \right\rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{I}_m^- &= \mathcal{I}_m^a - \mathcal{I}_m^b \end{aligned}$$

for $m = x, y$.

The two-well EPR-Steering criterion

Main result II

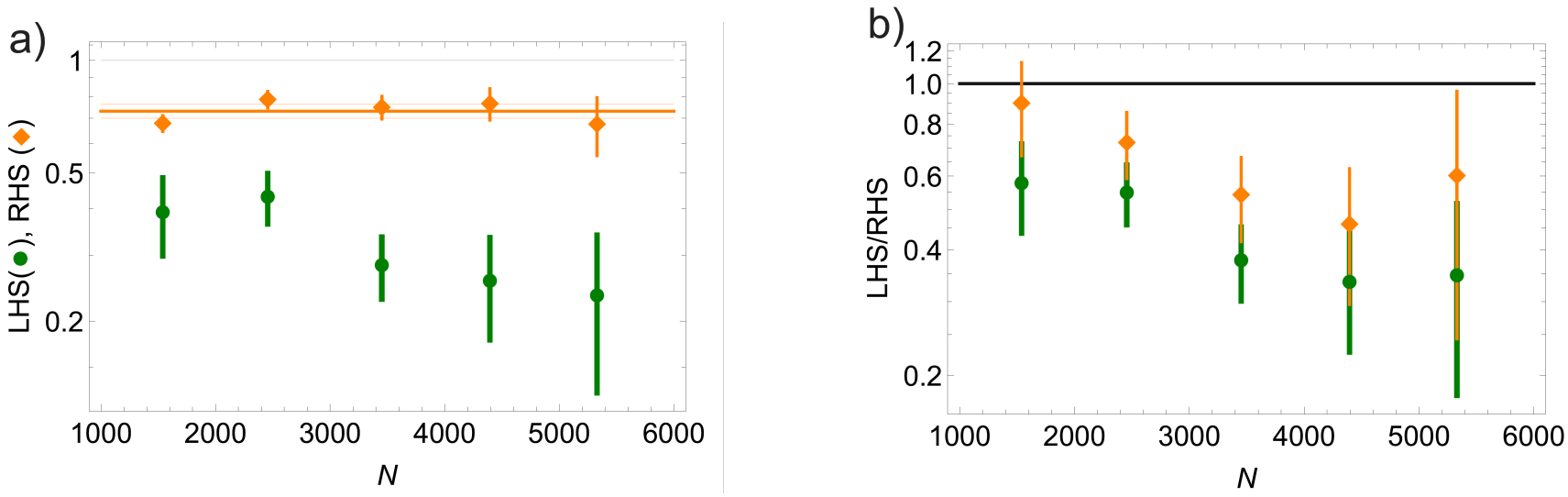
For states with a hidden state model,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{4}\left\langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \right\rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Violation of the criterion: entanglement is detected II



(a) LHS/RHS for Quantum 2023, and (b) for Science 2018.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann, B. Lücke, C. Klempt, GT, Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles, Quantum 7, 914 (2023).

K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Thanks to:

- Photonic experiments: N. Kiesel, W. Wieczorek, L. Knips, H. Weinfurter, MPQ München
- Singlet state experiments with cold atoms: R. J. Sewell, N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, M. W. Mitchell, ICFO, Barcelona
- Dicke state experiments with cold atoms: I. Kruse, J. Peise, K. Lange, B. Lücke, C. Klempt, Hannover U., Germany; J. Arlt, Arhus U., Denmark
- Theory: O. Gühne, Siegen U., Germany
- C. Knapp, H.-J. Briegel, Insbruck, Austria
- G. Vitagliano, IQOQI, Vienna, Austria
- I. Apellaniz, P. Hyllus, I. L. Egusquiza, I. Urizar-Lanz, UPV/EHU, Bilbao, Spain
- M. Fadel, ETH Zürich and Basel, Switzerland
- M. Kleinmann, Siegen U., Germany
- Z. Zimborás, Wigner, Hungary
- L. Santos, Hannover U., Germany
- L. Pezze, A. Smerzi, LENS, Firenze

Summary

- We discussed entanglement detection in particle ensembles close to singlet states and Dicke states.

- We also discussed the detection of bipartite entanglement in Dicke states.

THANK YOU FOR YOUR ATTENTION!

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