Detecting metrologically useful multiparticle entanglement of Dicke states

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Why multipartite entanglement is important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.

Outline

1. Introduction and motivation
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. Spin squeezing for Dicke states
   - Entanglement detection close to Dicke states
   - Detection of multipartite entanglement close to Dicke states
   - Experimental results
4. Detecting metrologically useful entanglement
   - Basics of quantum metrology
   - Metrology with measuring $\langle J_z^2 \rangle$
   - Metrology with measuring any operator

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

- Separable states remain separable under local operations. ("Local operations and classical communication")
- Separable states can be cerated without real quantum interaction. They are the "boring" states.

k-producibility/k-entanglement

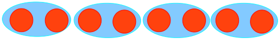
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

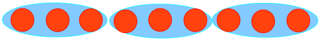
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.



two-producible



three-producible

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

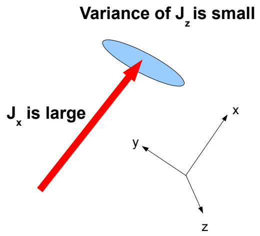
The standard spin-squeezing criterion

The spin squeezing criteria for entanglement detection is

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

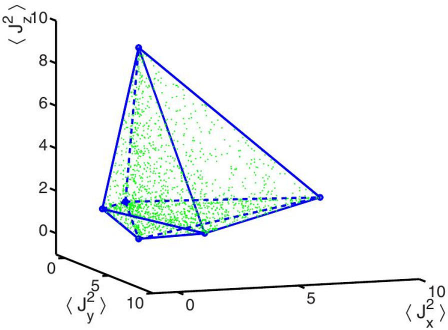
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke state)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]
[Singlets: Behbood *et al.*, Phys. Rev. Lett. 2014; GT, Mitchell, New. J. Phys. 2010.]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{\rm 2p}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{\rm 2p}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{\rm 2p} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{\rm 2p}$.
- Still, as we will see, we can even detect multipartite entanglement!

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009; Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Nat. Phys. 2012; C. Gross *et al.*, Nature 2011]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011; GT, PRA 2012; GT and Apellaniz, JPHYSA, 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2\rangle + \langle J_l^2\rangle - \tfrac{N}{2} \leq (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$

$$\langle J_z^2\rangle = 0.$$

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001);
experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

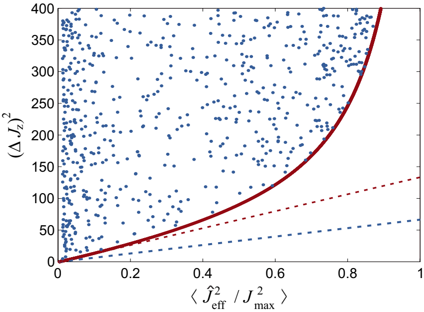
  which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Concrete example

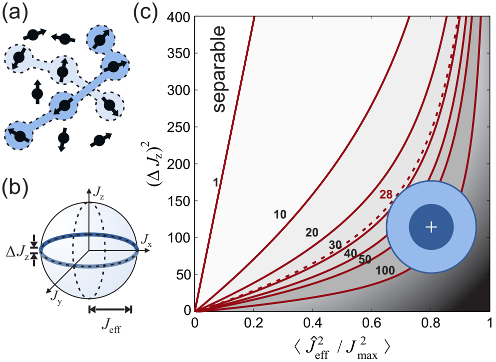


- $N = 8000$ particles, and $J_{\text{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- Red dashed line: tangent of our curve.

Outline

Experimental results

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



Giuseppe Vitagliano

[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

Diversion

Polish	English
Demkowicz-Dobrzański	Demkovi(ch)-Dob(zh)anski

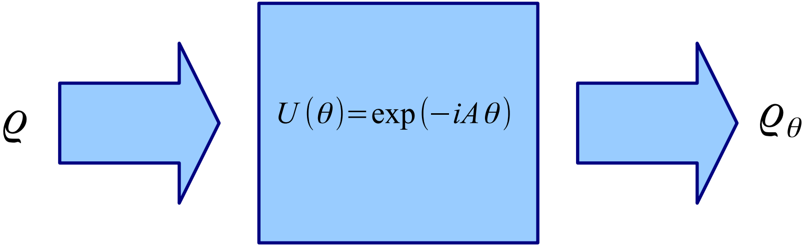
Other languages use different extra letters

Polish	Hungarian
Demkowicz-Dobrzański	Demkovi(cs)-Dob(zs)an(sz)ki

Outline

Quantum metrology

- Fundamental task in metrology



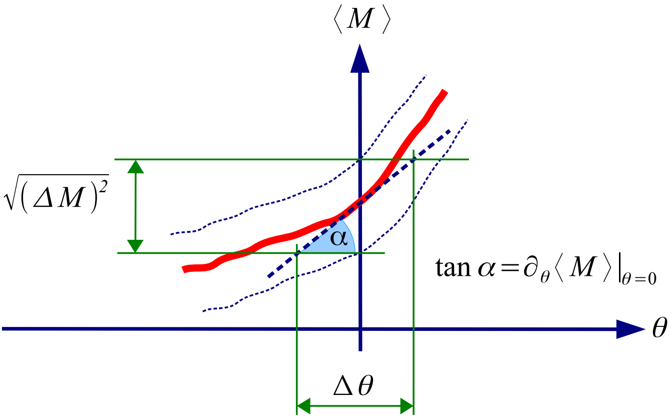
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- CramÃ©r-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \geq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \le N.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \le kN.$$

[Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2$$

[Fröwis, Dür, New J. Phys. 14 093039 (2012).]

Outline

Metrology with Dicke states

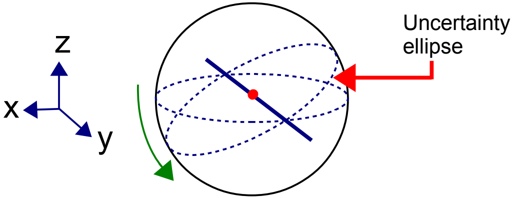
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)

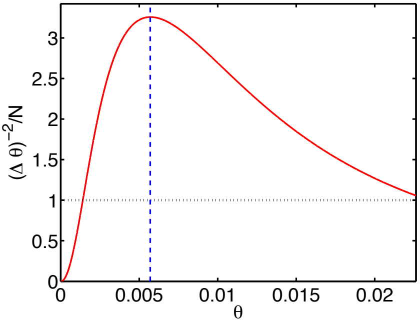


Metrology with Dicke states II

We measure $\langle J_z^2 \rangle$ to estimate θ. The precision is given by the error-propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta J_z^2)^2}{|\partial_\theta \langle J_z^2 \rangle|^2}.$$

- Precision as a function of θ for some noisy Dicke state [remember: $(\Delta\theta)^{-2} \geq F_Q[\varrho, A]$.]



Formula for maximal precision

Parameter value for the maximum

$$\tan^2 \theta_{\rm opt} = \sqrt{\frac{(\Delta J_z^2)^2}{(\Delta J_x^2)^2}}.$$

Consistency check: for the noiseless Dicke state we have $(\Delta J_z^2)^2 = 0$, hence $\theta_{\rm opt} = 0$.

Iagoba Apellaniz

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision II

Maximal precision with a closed formula

$$(\Delta\theta)^2_{\rm opt} = \frac{2\sqrt{(\Delta J_z^2)^2(\Delta J_x^2)^2}+4\langle J_x^2\rangle-3\langle J_y^2\rangle-2\langle J_z^2\rangle(1+\langle J_x^2\rangle)+6\langle J_z J_x^2 J_z\rangle}{4(\langle J_x^2\rangle-\langle J_z^2\rangle)^2}.$$

- Given in terms of collective observables, like spin-squeezing criteria.

[Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision III

- Some things are difficult to measure, they can be bounded

$$\langle J_z J_x^2 J_z \rangle = \frac{\langle J_z (J_x^2 + J_y^2) J_z \rangle}{2} = \frac{\langle J_z (J_x^2 + J_y^2 + J_z^2) J_z \rangle - \langle J_z^4 \rangle}{2} \leq \frac{N(N+2)}{8} \langle J_z^2 \rangle - \frac{1}{2} \langle J_z^4 \rangle.$$

- Equality holds for symmetric states.

[Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015).]

“Witnessing” the quantum Fisher information

- Important concept: we detect metrological usefulness without carrying out the metrological task.
- Advantages:
  - The experiment can be simpler, we do not need dynamics.
  - In principle, we could obtain $F_Q \propto N^2$ scaling even for large N.

[See Escher, de Matos Filho, Davidovich, Nat. Phys. 7, 406 (2011);
Demkowicz-Dobrzański, Kołodyński, Guţă, Nat. Commun. 3, 1063 (2012).]

Experimental test of our formula

- Trying the bound for the experimental data for $N = 7900$ particles

$$\langle J_z^2 \rangle = 112 \pm 31, \qquad \langle J_z^4 \rangle = 40 \times 10^3 \pm 22 \times 10^3,$$
$$\langle J_x^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6, \quad \langle J_x^4 \rangle = 6.2 \times 10^{13} \pm 0.8 \times 10^{13}.$$

- Hence, we obtain

$$\frac{(\Delta\theta)^{-2}_{\mathrm{opt}}}{N} \geq 3.7 \pm 1.5.$$

- Remember, for states for at most k-particle entanglement we have

$$(\Delta\theta)^{-2} \leq F_Q[\varrho, J_l] \leq kN.$$

- Thus, four-particle entanglement is detected for this particular measurement.

Interpretation of the results

k-particle multipartite entanglement is easier to get than *metrologically useful* k-particle multipartite entanglement.

- k-particle entangled state:

$$(k\,\text{particles}) \otimes [(k-1)\,\text{particles}] \otimes [(k-1)\,\text{particles}] \otimes \ldots$$

- Metrologically useful k-particle entangled state:

$$(k\,\text{particles}) \otimes |\text{GHZ}_{k-1}\rangle \otimes |\text{GHZ}_{k-1}\rangle \otimes \ldots,$$

where the state of $(k\,\text{particles})$ is better metrologically than $|\text{GHZ}_{k-1}\rangle$.

- Mixed states: we need detectable entanglement.

Outline

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, Petz, Phys. Rev. A 87, 032324 (2013); Yu, arXiv1302.5311 (2013);
GT, IApellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Let us put it in context

Due to convexity of F_Q and the concavity of the variance

$$\frac{1}{4}F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2{}_k \leq (\Delta A)^2{}_\varrho.$$

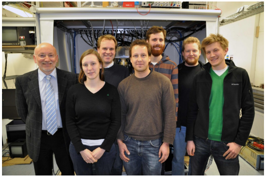
Both inequalities are tight.

We can also interpret this based on splitting the variance as to "quantum" and "classical" parts

$$(\Delta A)^2 = \sum_k p_k (\Delta A_k)^2 + (\langle A \rangle - \langle A \rangle_k)^2.$$

The quantum Fisher information is the minimal "quantum part" of the variance.

Project participants



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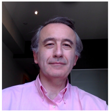

I.L. Egusquiza

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O. Gühne

Siegen

Summary

- Detection of multipartite entanglement and metrological usefulness close to Dicke states, by measuring collective quantities only. ☞ Talk of Iagoba Apellaniz

Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014).

Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt,
PRL 112, 155304 (2014)
(synopsis at physics.aps.org);

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);

Apellaniz, Kleinmann, Gühne, GT, arxiv: arXiv:1511.05203.

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