

# Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles

Quantum 7, 914 (2023)

G. Vitagliano<sup>1,2</sup>, M. Fadel<sup>3,4</sup>, I. Apellaniz<sup>2,5,6</sup>, M. Kleinmann<sup>7,2</sup>,  
B. Lücke<sup>8</sup>, C. Klempt<sup>8,9</sup>, G. Tóth<sup>2,5,10,11,12</sup>

<sup>1</sup>IQOQI, Wien, Austria, <sup>2</sup>University of the Basque Country UPV/EHU, Bilbao, Spain

<sup>3</sup>Department of Physics, ETH Zürich, Zürich, Switzerland

<sup>4</sup>Department of Physics, University of Basel, Basel, Switzerland

<sup>5</sup>EHU Quantum Center, University of the Basque Country UPV/EHU, Spain

<sup>6</sup>University of Mondragon, Spain, <sup>7</sup>University of Siegen, Germany

<sup>8</sup>Institut für Quantenoptik, Leibniz Universität Hannover, Hannover, Germany

<sup>9</sup>Deutsches Zentrum für Luft- und Raumfahrt e.V. (DLR)

<sup>10</sup>Donostia International Physics Center (DIPC), San Sebastián, Spain

<sup>11</sup>IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

<sup>12</sup>Wigner Research Centre for Physics, Budapest, Hungary

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea.
- In many cases, we need to detect **bipartite entanglement**.

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Entanglement

A state is **(fully) separable** if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \dots \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is **entangled**.

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Many-particle systems for $j=1/2$

- For spin- $\frac{1}{2}$  particles, we can measure the **collective angular momentum operators**:

$$\mathbf{J}_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where  $l = x, y, z$  and  $\sigma_l^{(k)}$  a Pauli spin matrices.

- We measure the **expectation values**  $\langle \mathbf{J}_l \rangle$ .
- We can also measure the **variances**

$$(\Delta \mathbf{J}_l)^2 := \langle \mathbf{J}_l^2 \rangle - \langle \mathbf{J}_l \rangle^2.$$

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions



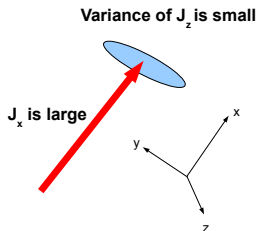
# The standard spin-squeezing criterion

The **spin squeezing criterion for entanglement detection** is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If  $\xi_s^2 < 1$  then the state is entangled.
- States detected are like this:



- They are good for metrology!

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \quad (\text{singlet states})$$

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \quad (\text{Dicke states})$$

$$(N-1) \left[ (\Delta J_k)^2 + (\Delta J_l)^2 \right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where  $k, l, m$  take all the possible permutations of  $x, y, z$ .

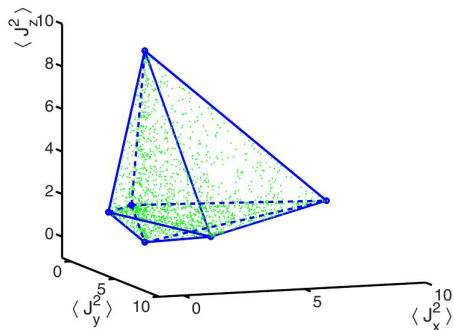
singlets: GT, Phys. Rev. A 69, 052327 (2004);

all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);

spin- $j$ : G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

# Generalized spin squeezing criteria for $j = \frac{1}{2} \mathbb{I}$

- Separable states are in the polytope



- We set  $\langle J_l \rangle = 0$  for  $l = x, y, z$ .

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Dicke states

- Dicke states: eigenstates of  $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$  and  $J_z$ .
- Symmetric Dicke states of spin-1/2 particles, with  $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- Summing over all permutations.
- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.*, PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al.*, Nat. Phys. 2012.

## Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;

GT, PRA 2012;

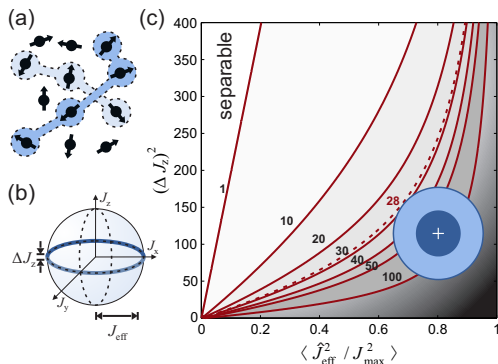
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... are macroscopically entangled, like GHZ states.

Fröwis, Dür, PRL 2011.

# Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J_{\text{eff}}^2 = J_x^2 + J_y^2, \quad J_{\text{max}} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).



# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

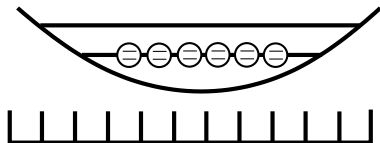
- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Bipartite entanglement from bosonic multipartite entanglement

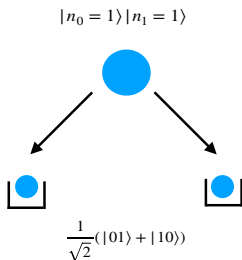
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

# Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument

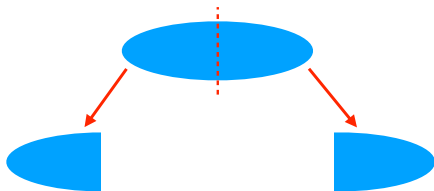


See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



# Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.
- **The splitting does not generate entanglement**, if we consider projecting to a fixed particle number.



# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in the spin state  $|j_z = 0\rangle$ .

- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

Tunneling from mode 0 to the mode +1 and -1.

- Two-particle example:

$$\begin{aligned} |j_z = 0\rangle |j_z = 0\rangle &\rightarrow \frac{1}{\sqrt{2}} (|j_z = +1\rangle |j_z = -1\rangle + |j_z = -1\rangle |j_z = +1\rangle) \\ &= \text{Dicke state of 2 particles.} \end{aligned}$$

# Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is,  $N - 2n$  particles remained in the  $|j_z = 0\rangle$  state, while  $2n$  particles form a symmetric Dicke state given as

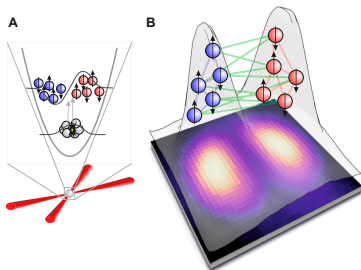
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

where we use  $|0\rangle$  and  $|1\rangle$  instead of  $|j_z = -1\rangle$  and  $|j_z = +1\rangle$ .

- Half of the atoms in state  $|0\rangle$ , half of the atoms in state  $|1\rangle$  + symmerization.

# Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



[ K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, *Science* 360, 416 (2018). ]



# Correlations for Dicke states

- For the Dicke state

$$(\Delta(J_x^a - J_x^b))^2 \approx 0,$$

$$(\Delta(J_y^a - J_y^b))^2 \approx 0,$$

$$(\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 = 0.$$

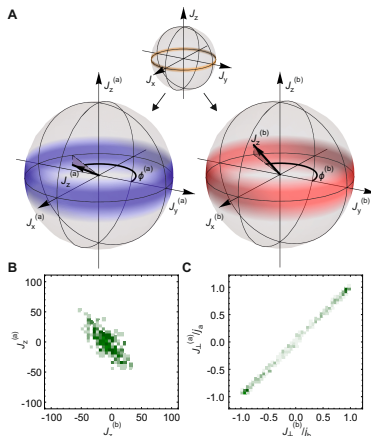
- Measurement results on well "b" can be predicted from measurements on "a"

$$J_x^b \approx J_x^a, \quad (\text{correlation})$$

$$J_y^b \approx J_y^a, \quad (\text{correlation})$$

$$J_z^b = -J_z^a. \quad (\text{anti-correlation})$$

# Correlations for Dicke states - experimental results



$$\text{Here, } J_{\perp}^{(n)} = \cos \alpha J_x^{(n)} + \sin \alpha J_y^{(n)}.$$

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2 [(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x \rangle^2 + \langle J_y \rangle^2).$$

Then,

$$(\Delta J_z)^2 [(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4} [(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x^2 \rangle + \langle J_y^2 \rangle).$$

- Simple algebra yields

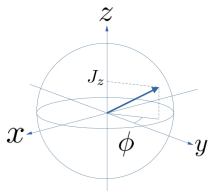
$$\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle} \geq \frac{1}{4}.$$

- Note that  $\langle J_x^2 \rangle$  appears, not  $\langle J_x \rangle^2$ .

# Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[ (\Delta J_z)^2 + \frac{1}{4} \right]}_{\sim \text{fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle}}_{\sim \text{phase fluctuation}} \geq \frac{1}{4}.$$



Handwaving description:

$J_z$  and  $\phi$  cannot be defined both with high accuracy.

# Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_x^n = \frac{J_x^n}{\sqrt{j_n(j_n + 1)}}, \quad \mathcal{J}_y^n = \frac{J_y^n}{\sqrt{j_n(j_n + 1)}},$$

where  $n = a, b$  (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables  $\rightarrow$  resistance to experimental imperfections.

# Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \left[ (\Delta \mathcal{J}_x)^2 + (\Delta \mathcal{J}_y)^2 \right] \geq \frac{1}{4} \langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \rangle.$$

# The two-well EPR-Steering criterion

## Main result I

For states with a hidden state model,

$$\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \left[ (\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2 \right] \geq \frac{1}{4} \langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$



# The two-well entanglement criterion

## Main result II

For separable states,

$$\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \left[ (\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2 \right] \geq \frac{1}{16} \langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$

# Outline

## 1 Motivation

- Why entanglement is important?

## 2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for  $j = \frac{1}{2}$

## 3 Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

## 4 Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Creating Dicke states in BEC
- Entanglement detection in Dicke states
- Our criteria work under realistic conditions

# Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Hence, the density matrix is of the form

$$\rho = \sum_{j_a, j_b} Q_{j_a, j_b} \rho_{j_a, j_b},$$

where  $Q_{j_a, j_b}$  are probabilities and  $\rho_{j_a, j_b}$  are states.

- Here  $j_a = N_a/2$ ,  $j_b = N_b/2$ .
- $\rho$  is entangled iff at least one of the  $\rho_{j_a, j_b}$  is entangled.

## Problem 1: Varying particle number II

- **Splitting noise**: Even if we have a constant total particle number, the ensemble will not be evenly split.
- Probability distribution for having  $N/2 + x$  particles

$$p_x = 2^{-N} \binom{N}{N/2 + x}.$$

- Variance

$$\text{var}(N_a) = \text{var}(x) = \langle x^2 \rangle = \frac{N}{4}.$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left( \frac{N}{8} + \frac{1}{2}x^2 \right) = \frac{N}{8} + \frac{1}{2}\text{var}(x) = \frac{N}{4}.$$

Twice as large due to the unequal splitting.

# Problem 1: Varying particle number III

- $N/2 : N/2$  splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}.$$

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

## Problem 1: Varying particle number IV

- **Solution:** Let use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a + 1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b + 1)}} J_l^b$$

for  $l = x, y$ .

- We obtain

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

- After splitting  $|x| \lesssim \sqrt{N/4}$ .
- We have

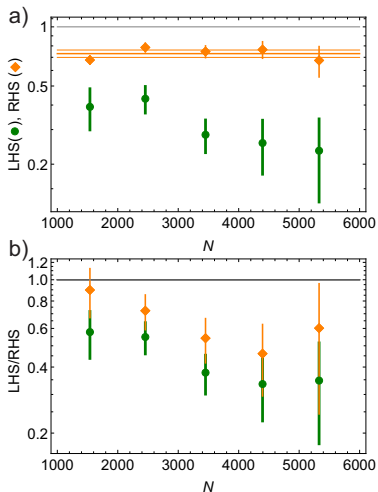
$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

$(\Delta \mathcal{J}_l^-)^2$  is not sensitive to the fluctuation of  $x$  if  $N$  is large.

## Problem 2: States are not always symmetric in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must handle this.

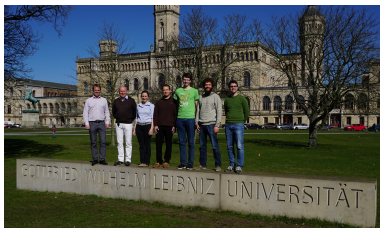
# Violation of the criterion: entanglement is detected II



LHS/RHS for (top) our present work, and (bottom) for Science 2018.



# Collaborators on entanglement conditions for double-well Dicke states



C. Klempt, I. Kruse, J. Peise,  
K. Lange, B. Lücke

**Hannover**



G. Vitagliano

**IQOQI, Wien**



I. Apellaniz

**Bilbao (G.T.)**



M. Fadel

**ETH Zürich, Basel**



M. Kleinmann

**U. of Siegen**

# Summary

- Detection of bipartite entanglement and EPR steering close to Dicke states. It works also for split spin-squeezed states.
- Non-symmetric states within the wells and a varying particle number can also be handled.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann,  
B. Lücke, C. Klempt, and G. Tóth,  
Number-phase uncertainty relations and bipartite entanglement  
detection in spin ensembles,  
[Quantum 7, 914 \(2023\)](#)

K. Lange *et al.*, [Science 334, 773–776 \(2011\)](#)

THANK YOU FOR YOUR ATTENTION!  
[www.gtoth.eu](http://www.gtoth.eu)