Entanglement between two spatially separated atomic modes

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Tromsø
Murmansk
Kiruna
Norwegian Sea
White Sea
Arkhangelsk
Reykjavík
ICELAND
Arctic Circle
Luleå
Oulu
NORWAY
Umeå
FINLAND
Lake Onega
Trondheim
SWEDEN
Tórshavn
Faroe Islands (DENMARK)
Gulf of Bothnia
Lake Ladoga
Tampere
Helsinki
Turku
St. Petersburg
SHETLAND ISLANDS
Gävle
Bergen
Oslo
Stockholm
ÅLAND ISLANDS
Gulf of Finland
Tallinn
ESTONIA
RUSSIA
Rockall (U.K.)
ORKNEY ISLANDS
HEBRIDES
Stavanger
Skagerrak
Kattegat
Gotland
LATVIA
Riga
North Atlantic Ocean
Aberdeen
Glasgow
Edinburgh
UNITED
North Sea
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
DENMARK
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RUSSIA
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Mahilyow
Dublin
Irish Sea
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Gdańsk
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SAN MARINO
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Andorra la Vella
Marseille
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GREECE
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Sevilla
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Cagliari
Ionian Sea
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Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
0 300 Kilometers
0 20 300 Miles

Outline

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \ldots \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

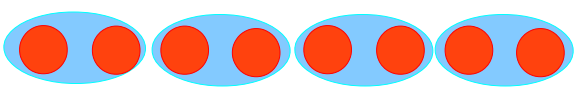
A pure state is k-producible if it can be written as

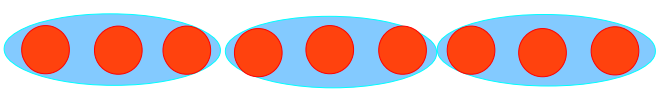
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.





Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \tfrac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

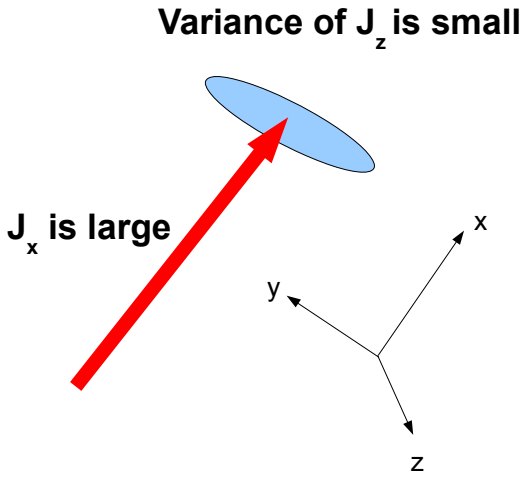
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_S^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_S^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

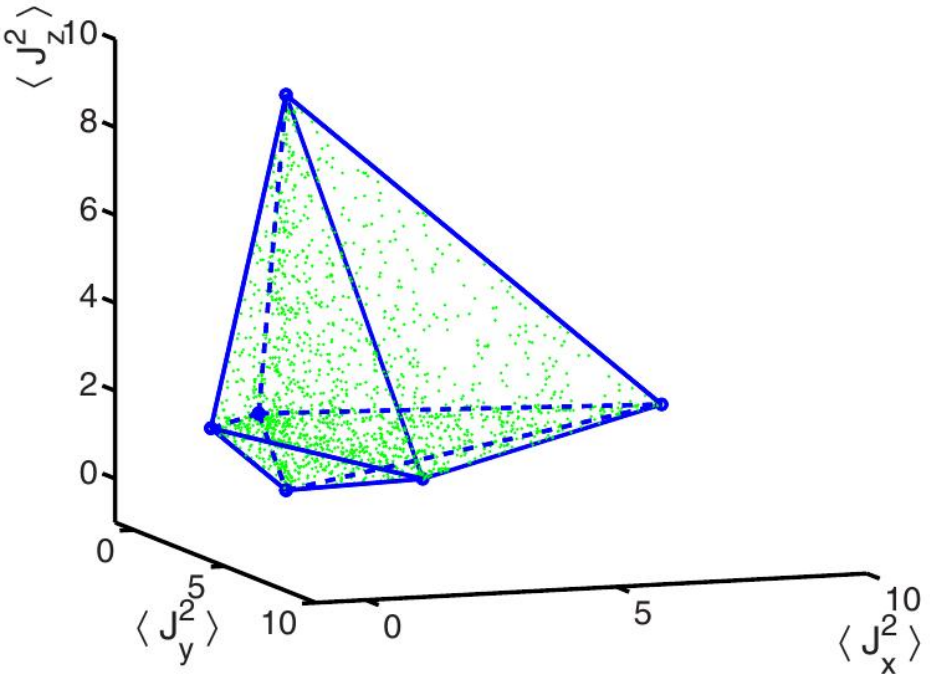
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can detect states with a separable $\varrho_{2\mathrm{p}}$.
- We can even detect multipartite entanglement!

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.
- Symmetric Dicke states with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.*, PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.]

[cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for "50 year of Bell's theorem", 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011.]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

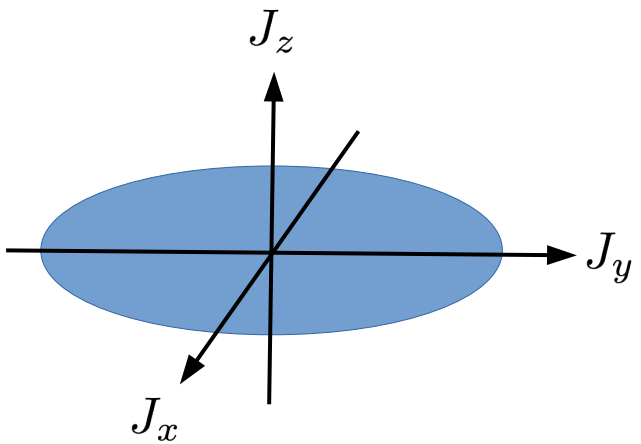
holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$

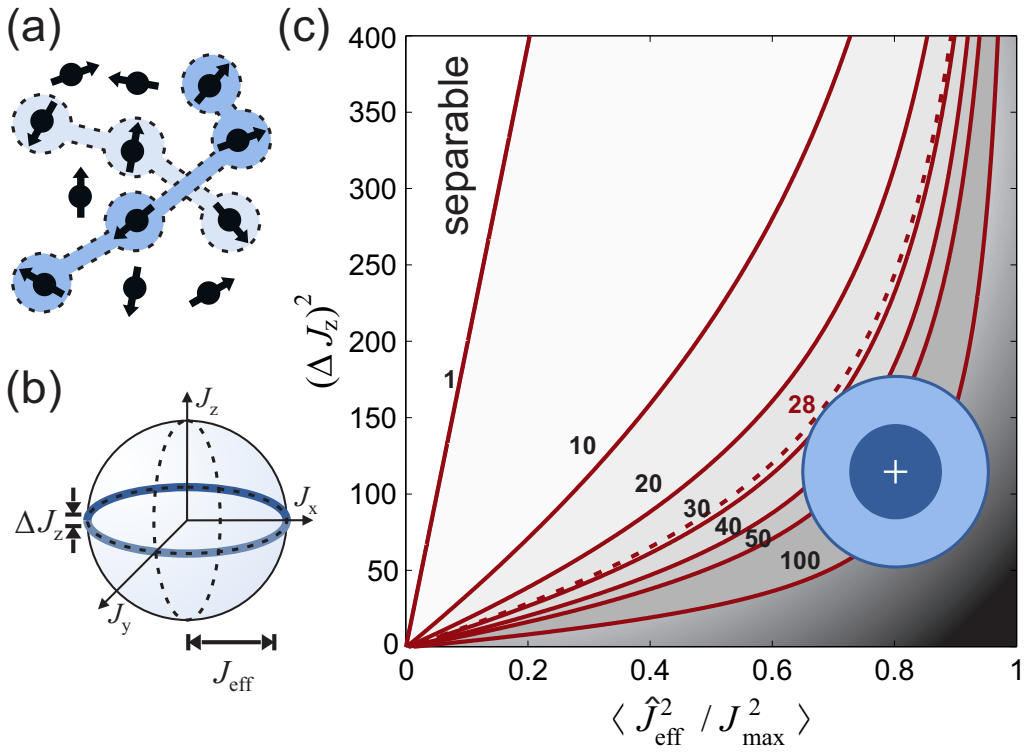
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\mathrm{eff}} = J^2_x + J^2_y, \quad J_{\mathrm{max}} = \frac{N}{2}.$$

[Lücke *et al.*, PRL 112, 155304 (2014).]

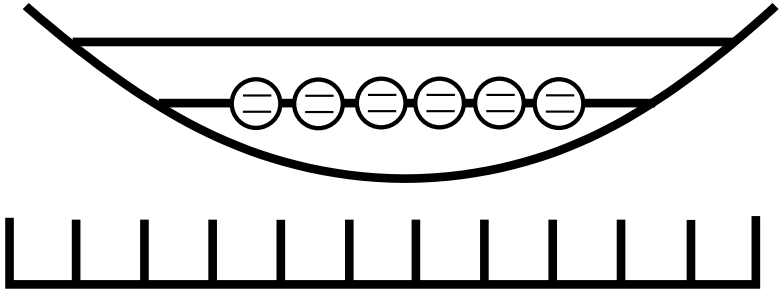
Outline

Bipartite entanglement from bosonic multipartite entanglement

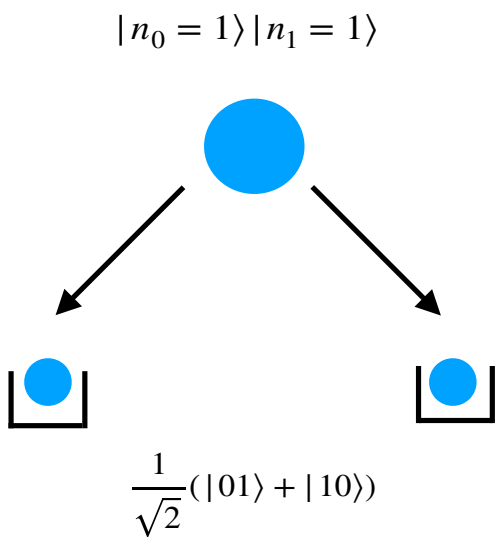
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument

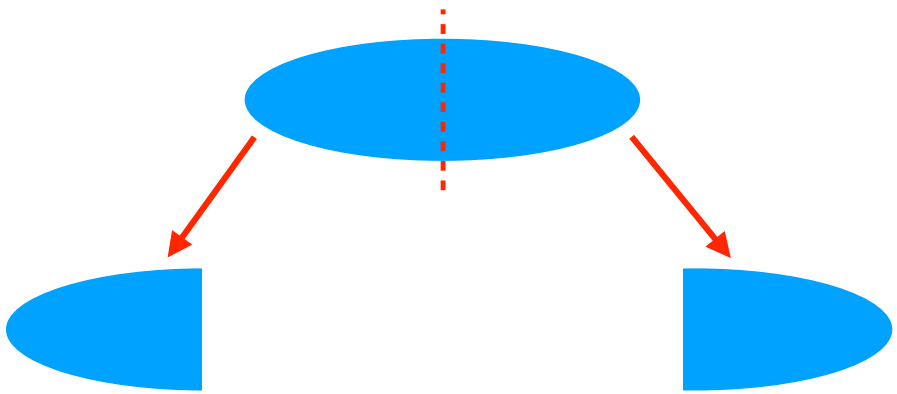


[See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)]



Bipartite entanglement from bosonic multipartite entanglement III

- Splitting of the ensembles: after splitting into two, we have bipartite entanglement if we had before multipartite entanglement.
- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



[N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).]

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in state $|0\rangle$.
- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

  Tunneling from mode 0 to the mode $+1$ and -1.

- Understanding the tunneling process

$$|00\rangle \quad \rightarrow \quad \frac{1}{\sqrt{2}}(|1,-1\rangle + |-1,1\rangle) = \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

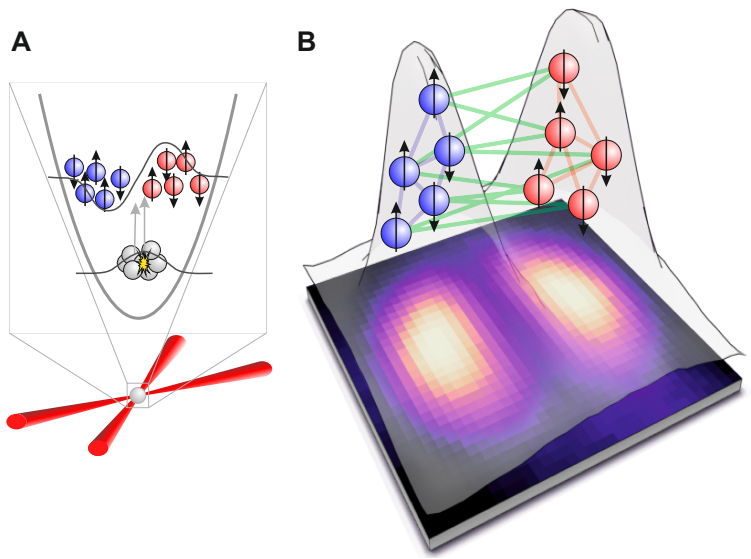
- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the 0 state, while $2n$ particles form a symmetric Dicke state.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



[K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).]

Very simple entanglement criterion for singlets

- For separable states of two large spins

$$[\Delta(J_{\mathrm{x}}^{(\mathrm{a})} + J_{\mathrm{x}}^{(\mathrm{b})})]^2 + [\Delta(J_{\mathrm{y}}^{(\mathrm{a})} + J_{\mathrm{y}}^{(\mathrm{b})})]^2 + [\Delta(J_{\mathrm{z}}^{(\mathrm{a})} + J_{\mathrm{z}}^{(\mathrm{b})})]^2 \geq \frac{N}{2}$$

  hold. For singlets, the LHS is zero.

- *Proof.* For product states $|\Psi_{\mathrm{a}}\rangle \otimes |\Psi_{\mathrm{b}}\rangle$

$$\sum_{m=\mathrm{x,y,z}} [\Delta(J_m^{(\mathrm{a})} + J_m^{(\mathrm{b})})]^2 = \sum_{m=\mathrm{x,y,z}} (\Delta J_m^{(\mathrm{a})})^2 + \sum_{m=\mathrm{x,y,z}} (\Delta J_m^{(\mathrm{a})})^2 \geq \frac{N_{\mathrm{a}}}{2} + \frac{N_{\mathrm{b}}}{2}.$$

  holds.

- True also for separable states due to the concavity of the variance.
  [GT, Phys. Rev. A (2004).]

Very simple entanglement criterion for Dicke states

- For separable states of two large spins

$$[\Delta(J_x^{(a)} - J_x^{(b)})]^2 + [\Delta(J_y^{(a)} - J_y^{(b)})]^2 + [\Delta(J_z^{(a)} + J_z^{(b)})]^2 \geq \frac{N}{2}.$$

- For Dicke states, the LHS is around $\frac{N}{4}$ for large N, since

$$\begin{aligned} [\Delta(J_z^{(a)} + J_z^{(b)})]^2 &= 0, \\ [\Delta(J_m^{(a)} + J_m^{(b)})]^2 &= \text{large}, \\ [\Delta(J_m^{(a)} - J_m^{(b)})]^2 &\approx \frac{N}{8} = \text{small} \end{aligned}$$

  for $m = x, y$.

- Not a practical criterion since small noise makes the state undetectable, and it assumes symmetry.

Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2 + \left\langle J_y\right\rangle^2).$$

  Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle + \left\langle J_y^2\right\rangle).$$

- Simple algebra yields

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

Normalized variables

- Let us introduce the normalized variables

$$\tilde{J}_m^{(n)} = \frac{J_m^{(n)}/j_n}{\mathcal{J}^{(n)}},$$

where $m = x, y$ and $n = \mathrm{a}, \mathrm{b}$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

and

$$\mathcal{J}^{(n)} = \left\langle \frac{(J_{\mathrm{x}}^{(n)})^2 + (J_{\mathrm{y}}^{(n)})^2}{j_n^2} \right\rangle^{\frac{1}{2}}.$$

- $\mathcal{J}^{(n)} \approx 1$ indicates a state close to be symmetric in the well, which is the case ideally. In general, $\mathcal{J}^{(n)} \leq 1$.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \tilde{J}_x)^2 + (\Delta \tilde{J}_y)^2\right] \geq \frac{1}{4}.$$

We define

$$\begin{aligned} J_z^+ &= J_z^{(a)} + J_z^{(b)}, \\ \tilde{J}_m^- &= \tilde{J}_m^{(a)} - \tilde{J}_m^{(b)} \end{aligned}$$

for $m = \mathrm{x}, \mathrm{y}$.

The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result

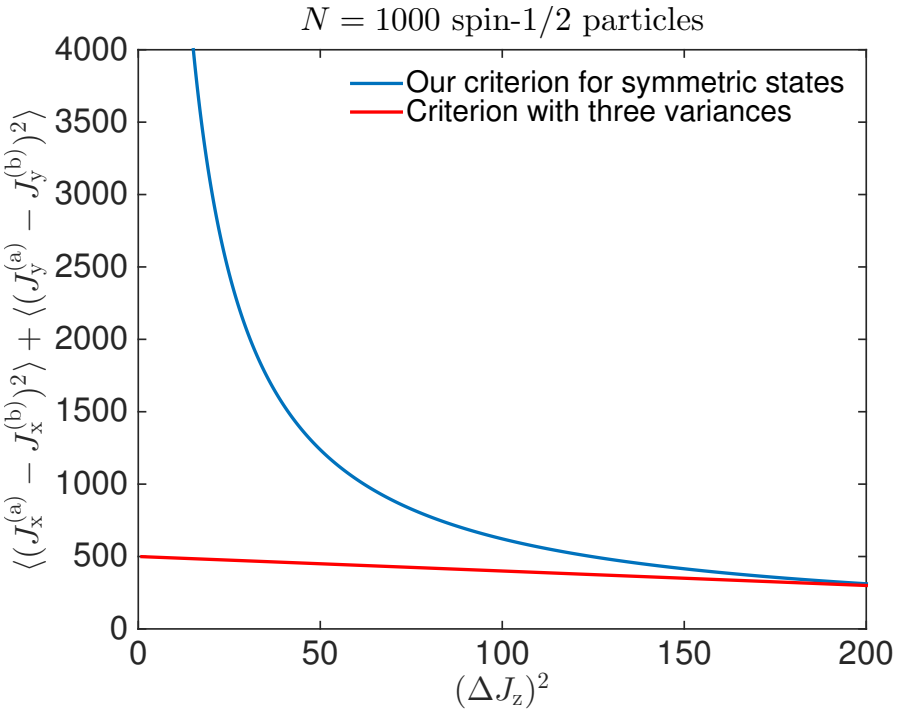
For separable states,

$$\left[(\Delta J_{\mathrm{z}}^{+})^2+\frac{1}{2}\right]\times\left[\langle(\tilde{J}_{\mathrm{x}}^{-})^2+(\tilde{J}_{\mathrm{y}}^{-})^2\rangle\right]\geq f\left(\mathcal{I}^{(\mathrm{a})},\mathcal{I}^{(\mathrm{b})}\right)$$

holds, where $f(x,y)=\frac{(x^2+y^2-1)^2}{xy}$,

Any state violating the inequality is entangled.

Our entanglement criterion vs. simple criterion



- For $j_a = j_b = \frac{N}{4}$ and symmetric states in the wells

$$\left[(\Delta J_z)^2 + \frac{1}{2}\right]\left[(\Delta(J_x^{(a)} - J_x^{(b)}))^2 + (\Delta(J_y^{(a)} - J_y^{(b)}))^2\right] \geq \frac{N^2}{16}.$$

Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Consider a density matrix

$$\varrho = \sum_{j_\mathrm{a}, j_\mathrm{b}} Q_{j_\mathrm{a}, j_\mathrm{b}} \varrho_{j_\mathrm{a}, j_\mathrm{b}},$$

  where $\varrho_{j_\mathrm{a}, j_\mathrm{b}}$ are states with $2j_\mathrm{a}$ and $2j_\mathrm{b}$ particles in the two wells, $Q_{j_\mathrm{a}, j_\mathrm{b}}$ are probabilities.

- ϱ is entangled iff at least one of the $\varrho_{j_\mathrm{a}, j_\mathrm{b}}$ is entangled.
- We use special normalization in the criterion.

Problem 2: States are not always symmertic in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must hadle this.

Correlations for Dicke states

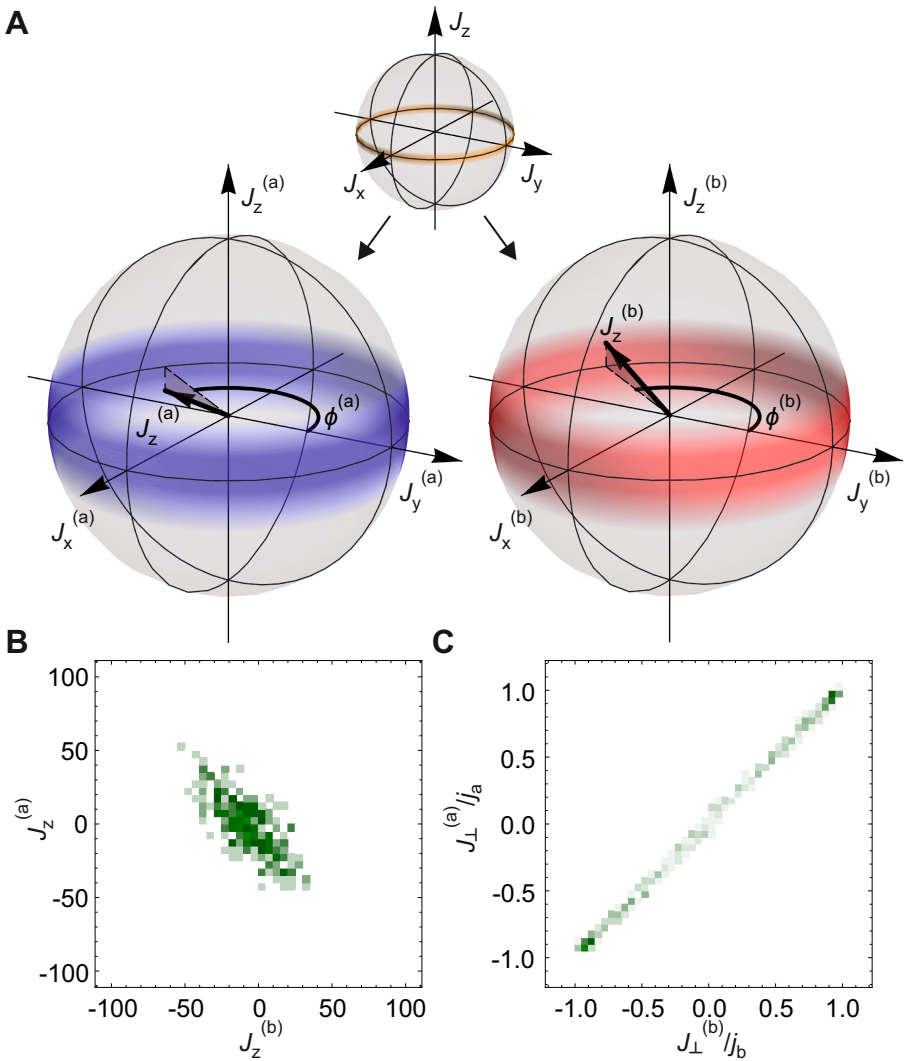
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_x^{(a)} - J_x^{(b)}))^2 &\approx 0, \\
(\Delta(J_y^{(a)} - J_y^{(b)}))^2 &\approx 0, \\
(\Delta J_z)^2 &= 0.
\end{aligned}
$$

- Measurement results on well "b" can be predicted from measurements on "a"

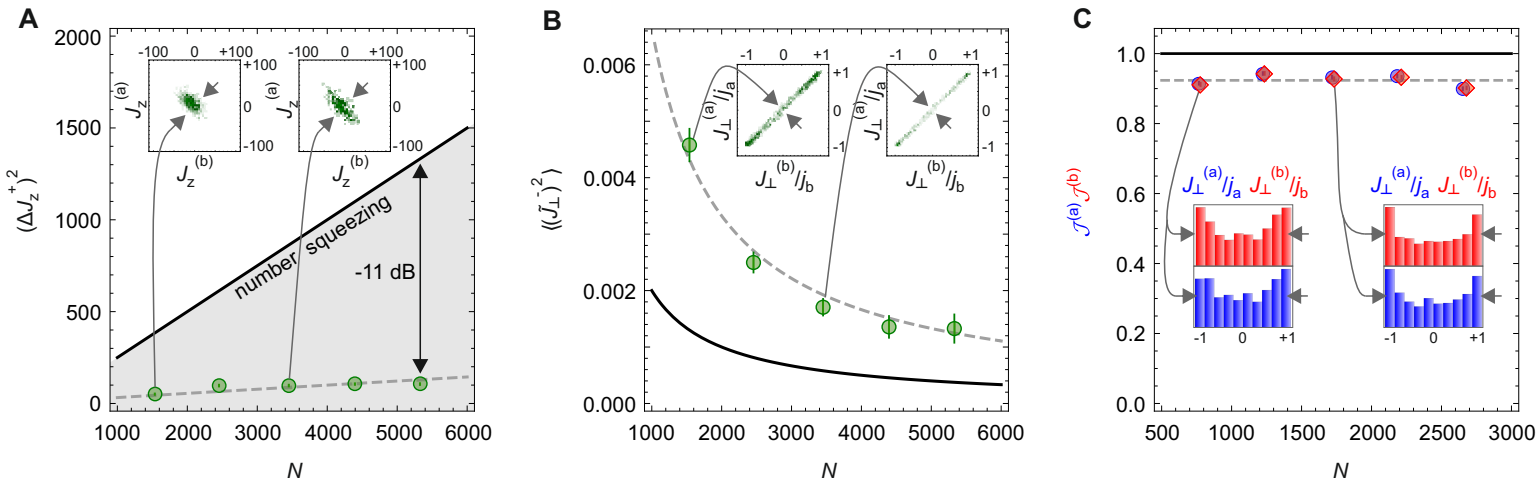
$$
\begin{aligned}
J_x^{(b)} &\approx J_x^{(a)}, \\
J_y^{(b)} &\approx J_y^{(a)}, \\
J_z^{(b)} &= -J_z^{(a)}.
\end{aligned}
$$

Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Further experimental results



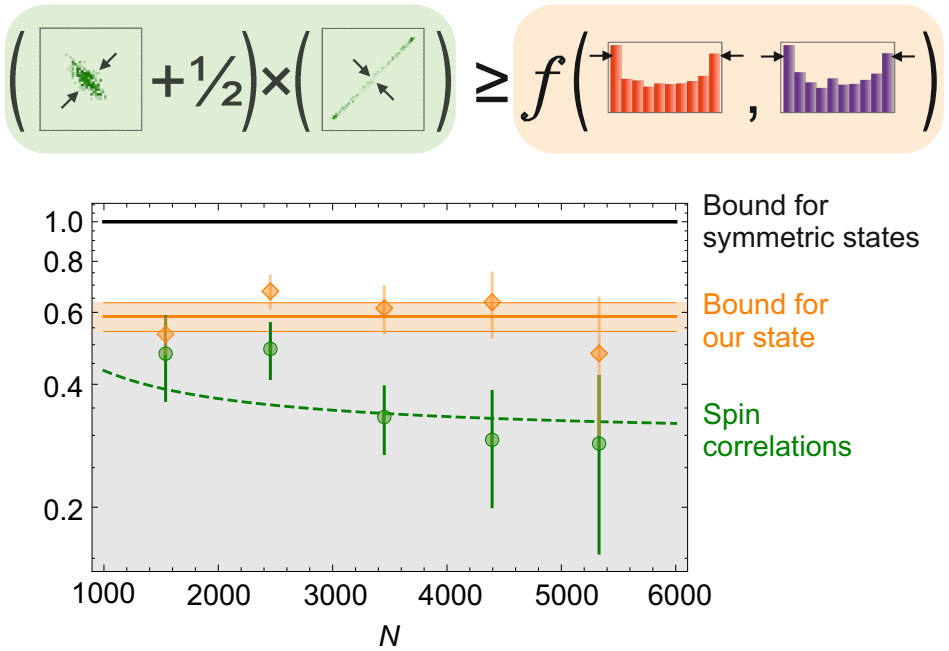
A. Black: shot-noise limit. Green circles: experiments.

B: Black: shot-noise limit. Green circles: experiments.

C: Black: perfect symmetry. Blue/red: values for the left/right well.

$$\mathcal{J}^{(n)} = \left\langle \frac{(J_{\mathrm{x}}^{(n)})^2 + (J_{\mathrm{y}}^{(n)})^2}{j_n^2} \right\rangle^{\frac{1}{2}} .$$

Violation of the criterion: entanglement is detected

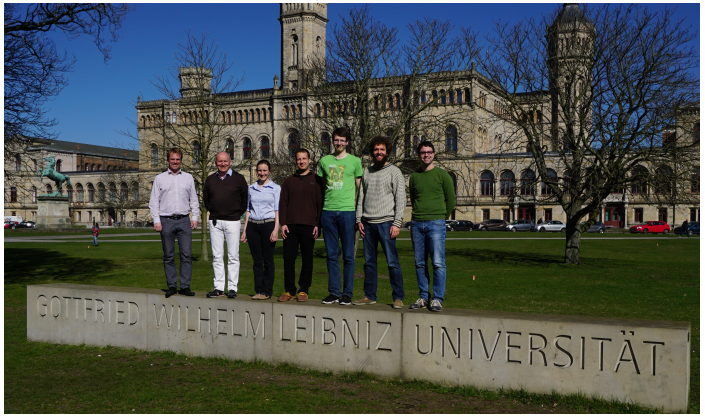


For separable states,

$$\left[(\Delta J_z^+)^2 + \frac{1}{2}\right] \times \left[\langle(\tilde{J}_x^-)^2 + (\tilde{J}_y^-)^2\rangle\right] \geq f\left(\mathcal{J}^{(a)}, \mathcal{J}^{(b)}\right)$$

holds, where $f(x,y) = \frac{(x^2+y^2-1)^2}{xy}$.

Collaborators on entanglement conditions for Dicke states



C. Klempt, I. Kruse, J. Peise,
K. Lange, B. Lücke

G. Vitagliano

I. Apellaniz

M. Kleinmann

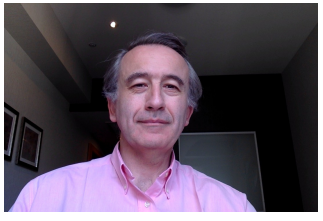
P. Hyllus

L. Santos

Hannover

O. Gühne

Siegen



I.L. Egusquiza

Bilbao (G.T.)

Summary

- Detection of bipartite entanglement close to Dicke states.
- Non-symmetric states within the wells and a varying particle number can also be handled.

K. Lange, J. Peise, B. Lücke, I. Kruse,
G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt,
Entanglement between two spatially separated atomic modes,
Science 360, 416 (2018).

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Appendix

Proof

Product states. For states of the form $|\Psi^{(\mathrm{a})}\rangle \otimes |\Psi^{(\mathrm{b})}\rangle$.

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[(\Delta \tilde{J}_{\mathrm{x}}^{-})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{-})^2\right]$$
$$= [(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4}) + (\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})] \cdot (\mathcal{V}^{(\mathrm{a})} + \mathcal{V}^{(\mathrm{b})})$$
$$\geq 4\sqrt{(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4})(\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})\mathcal{V}^{(\mathrm{a})}\mathcal{V}^{(\mathrm{b})}} \geq 1$$

holds, where we used the notation

$$\mathcal{U}^{(n)} = (\Delta J_{\mathrm{z}}^{(n)})^2, \qquad \mathcal{V}^{(n)} = (\Delta \tilde{J}_{\mathrm{x}}^{(n)})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{(n)})^2$$

for $n = \mathrm{a}, \mathrm{b}$. We used that

(i) $[\Delta(A^{(\mathrm{a})} + A^{(\mathrm{b})})]^2 = (\Delta A^{(\mathrm{a})})^2 + (\Delta A^{(\mathrm{b})})^2$,
(ii) Inequality between the arithmetic and the geometric mean,
(iii) Our number-phase like uncertainty.

Proof II

Using $\langle(\tilde{J}_{\mathrm{x}}^{(n)})^2\rangle + \langle(\tilde{J}_{\mathrm{y}}^{(n)})^2\rangle = 1$ for $n = \mathrm{a}, \mathrm{b}$, our inequality for product states yields

$$2\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq \mathcal{S},$$

where correlations between the two subsystems are characterized by

$$\mathcal{C} = \left\langle \frac{J_{\mathrm{x}}^{(\mathrm{a})} J_{\mathrm{x}}^{(\mathrm{b})} + J_{\mathrm{y}}^{(\mathrm{a})} J_{\mathrm{y}}^{(\mathrm{b})}}{j_{\mathrm{a}} j_{\mathrm{b}}} \right\rangle,$$

and

$$\mathcal{S} = \mathcal{J}^{(\mathrm{a})} \mathcal{J}^{(\mathrm{b})}.$$

$\mathcal{C}$ can be negative and $|\mathcal{C}| \leq \mathcal{S}$.

The normalization with the total spin will make it easier to adapt our criterion to experiments with a varying particle number in the ensembles.

Proof III

Separable states. We now consider a mixed separable state of the form $\varrho_{\rm sep} = \sum_k p_k |\Psi_k^{(a)}\rangle \otimes |\Psi_k^{(b)}\rangle$. For such states, we can write the following series of inequalities

$$2\left[(\Delta J_z^+)^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq 2\left[\sum_k p_k (\Delta J_z)^2{}_k + \frac{1}{2}\right]\left[\sum_k p_k \left(\mathcal{S}_k - \mathcal{C}_k\right)\right]$$

$$\geq 2\left[\sum_k p_k \sqrt{\left((\Delta J_z)^2{}_k + \frac{1}{2}\right)(\mathcal{S}_k - \mathcal{C}_k)}\right]^2 \geq \left(\sum_k p_k \sqrt{\mathcal{S}_k}\right)^2,$$

Subscript k refers to the $k^{\rm th}$ sub-ensemble $|\Psi_k^{(a)}\rangle \otimes |\Psi_k^{(b)}\rangle$.
(i) The first inequality in is due to $(\Delta J_z^+)^2$ and $\mathcal{S}$ being concave in the quantum state.
(ii) The second inequality is based on the Cauchy-Schwarz inequality.
(iii) The third inequality is the application of the previous inequality for all sub-ensembles.

Proof IV

Next, we find a lower bound on the RHS of the last inequality based on the knowledge of $\mathcal{I}^{(a)}$ and $\mathcal{I}^{(b)}$. We find that

$$\sum_k p_k \left(\mathcal{I}_k^{(a)} \mathcal{I}_k^{(b)}\right)^{1/2} \geq (\mathcal{I}^{(a)})^2 + (\mathcal{I}^{(b)})^2 - 1,$$

which is based on noting $(xy)^{1/4} \geq x + y - 1$ for $0 \leq x, y \leq 1$.

Using this to bound the RHS from below and dividing by $\mathcal{S}$ we obtain

$$\left[(\Delta J_z^+)^2 + \frac{1}{2}\right] \times \left[2 - 2\frac{C}{\mathcal{S}}\right] \geq \frac{\left[(\mathcal{I}^{(a)})^2 + (\mathcal{I}^{(b)})^2 - 1\right]^2}{\mathcal{S}}.$$