Entanglement criterion for two spatially separated atomic ensembles

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
HEBRIDES
Rockall (U.K.)
North Atlantic Ocean
North Sea
UNITED KINGDOM
IRELAND
Dublin
Belfast
Glasgow
Edinburgh
Aberdeen
Leeds
Liverpool
Manchester
Birmingham
Cardiff
London
Celtic Sea
English Channel
Guernsey (U.K.)
Jersey (U.K.)
Bergen
Oslo
Stavanger
Stockholm
Göteborg
Trondheim
Hammerfest
Tromsø
Kiruna
Luleå
Umeå
Gävle
Oulu
Tampere
Helsinki
Turku
ALAND ISLANDS
Murmansk
White Sea
Arkhangel
Lake Onega
Lake Ladoga
St. Petersburg
Gulf of Bothnia
Gulf of Finland
Tallinn
ESTONIA
Riga
LATVIA
LITHUANIA
Vilnius
Gotland
Baltic Sea
Öland
Kaliningrad
RUSSIA
Bornholm
DENMARK
Copenhagen
Malmö
Vitsyebsk
Smolensk
Minsk
Mahilyow
BELARUS
Hrodna
Homyel
Brest
Gdańsk
POLAND
Warsaw
Poznań
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ANDORRA
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PORTUGAL
Porto
Lisbon
Madrid
SPAIN
Valencia
Sevilla
Málaga
Gibraltar (U.K.)
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Rabat
Casablanca
MOROCCO
Algiers
ALGERIA
Tunis
TUNISIA
Balearic Sea
BALEARIC ISLANDS
Mediterranean Sea
Corsica
Sardinia
Cagliari
Ligurian Sea
Turin
Milan
Genoa
Venice
SAN MARINO
Florence
ITALY
Rome
VATICAN CITY
Naples
Tyrrhenian Sea
Palermo
Sicily
Valletta
MALTA
Adriatic Sea
Ionian Sea
SLOVENIA
Ljubljana
Zagreb
CROATIA
BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
SERBIA AND MONTENEGRO
Tirana
ALB.
Skopje
F.Y.R.O.M.
Thessaloníki
GREECE
Athens
Aegean Sea
Crete
Rhodes
Sofia
BULGARIA
Varna
Istanbul
ROMANIA
Cluj-Napoca
Bucharest
Constanța
Iași
MOLDOVA
Chisinau
Chernivtsi
L'viv
Rivne
UKRAINE
Kiev
Mykolayiv
Danube
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Dicke states**
   - Detection of multipartite entanglement close to Dicke states
   - Dicke state in double well

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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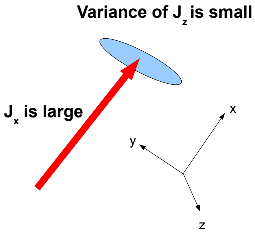
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_{\mathrm{x}} \rangle, \langle J_{\mathrm{y}} \rangle, \langle J_{\mathrm{z}} \rangle),$$
$$\vec{K} := (\langle J_{\mathrm{x}}^2 \rangle, \langle J_{\mathrm{y}}^2 \rangle, \langle J_{\mathrm{z}}^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

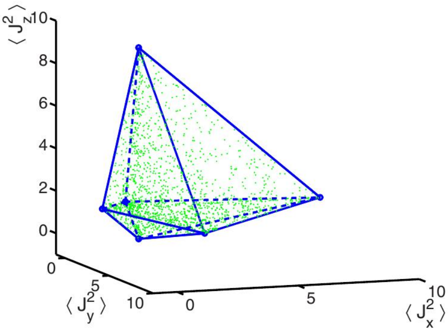
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \le \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \le (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \ge \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{2\mathrm{p}}$.
- Still, as we will see, we can even detect multipartite entanglement!

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Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states with $\langle J_z \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel *et al.,* PRL 2007; Wieczorek *et al.*, PRL 2009; Prevedel *et al.*, PRL 2009.]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Nat. Phys. 2012.]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011.]

[GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for "50 year of Bell's theorem", 2014.]

- ... are macroscopically entangled, yes, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

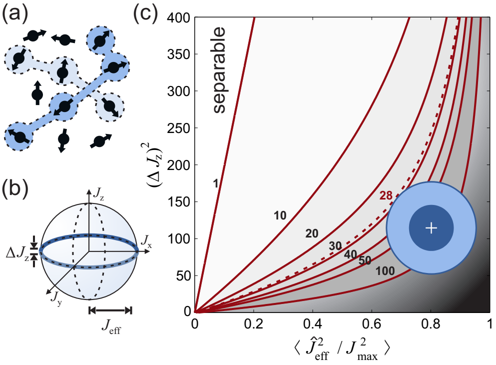
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$
$$\langle J_z^2 \rangle = 0.$$

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\text{eff}} = J_x^2 + J_y^2, \quad J_{\max} = \frac{N}{2}.$$

[Lücke *et al.*, PRL 112, 155304 (2014).]

Metrology with Dicke states

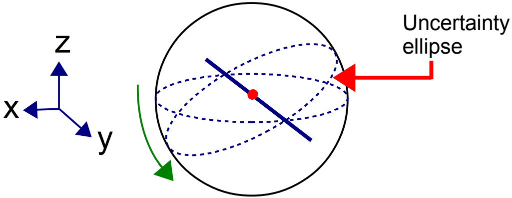
- For the Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Rotate the "pancake" and estimate the angle.



[B. Lücke *et al.*, Science 2011;
I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, NJP 2015.]

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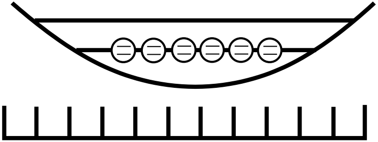
Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

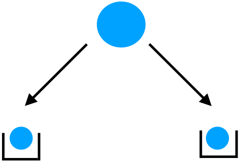
Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



[See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)]

$|n_0 = 1\rangle |n_1 = 1\rangle$

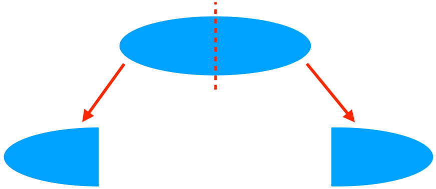


$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Bipartite entanglement from bosonic multipartite entanglement III

- Splitting of the ensembles: after splitting into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



[N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).]

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in state $|0\rangle$.
- Dynamics
$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$
Tunneling from mode 0 to the mode $+1$ and -1.
- In the particle picture
$$|00\rangle \to \frac{1}{\sqrt{2}}(|1,-1\rangle + |-1,1\rangle).$$

Experiment in the group of Carsten Klempt at the University of Hannover II

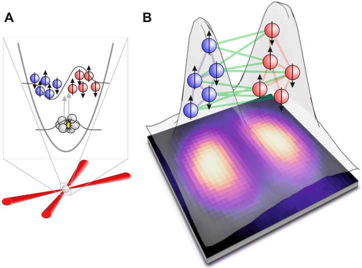
- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- Equivalently, $N - 2n$ particles remained in the 0 state, while $2n$ particles form a symmetric Dicke state.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important point: they used the first excited spatial mode of the trap, not the ground state mode.
- It has two "bumps" rather than one: thus they obtained a split Dicke state.



[K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).]

Very simple entanglement criterion for singlets

- For separable states

$$[\Delta(J_{\rm x}^{(\rm a)} + J_{\rm x}^{(\rm b)})]^2 + [\Delta(J_{\rm y}^{(\rm a)} + J_{\rm y}^{(\rm b)})]^2 + [\Delta(J_{\rm z}^{(\rm a)} + J_{\rm z}^{(\rm b)})]^2 \geq \frac{N}{2}.$$

  For singlets, the LHS is zero.

- *Proof.* For product states $|\Psi_{\rm a}\rangle \otimes |\Psi_{\rm b}\rangle$

$$\sum_{m={\rm x,y,z}} [\Delta(J_m^{(\rm a)} + J_m^{(\rm b)})]^2 = \sum_{m={\rm x,y,z}} (\Delta J_m^{(\rm a)})^2 + \sum_{m={\rm x,y,z}} (\Delta J_m^{(\rm a)})^2 \geq \frac{N_{\rm a}}{2} + \frac{N_{\rm b}}{2}.$$

  holds.

- True also for separable states due to the concavity of the variance.
  [GT, Phys. Rev. A (2004).]

Very simple entanglement criterion for Dicke states

- For separable states of two large spins

$$[\Delta(J_{\mathrm{x}}^{(\mathrm{a})}-J_{\mathrm{x}}^{(\mathrm{b})})]^2+[\Delta(J_{\mathrm{y}}^{(\mathrm{a})}-J_{\mathrm{y}}^{(\mathrm{b})})]^2+[\Delta(J_{\mathrm{z}}^{(\mathrm{a})}+J_{\mathrm{z}}^{(\mathrm{b})})]^2\geq\frac{N}{2}.$$

  For Dicke states, the LHS is around $\frac{N}{4}$ for large N, since

$$\begin{aligned}[\Delta(J_{\mathrm{z}}^{(\mathrm{a})}+J_{\mathrm{z}}^{(\mathrm{b})})]^2 &= 0,\\ [\Delta(J_{\mathrm{m}}^{(\mathrm{a})}+J_{\mathrm{m}}^{(\mathrm{b})})]^2 &= \mathrm{large},\\ [\Delta(J_{\mathrm{m}}^{(\mathrm{a})}-J_{\mathrm{m}}^{(\mathrm{b})})]^2 &\approx \frac{N}{8}=\mathrm{small}\end{aligned}$$

  for $m=\mathrm{x},\mathrm{y}$.

- Not a practical criterion since small noise makes the state undetectable, and it assumes symmetry.

Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2 + \left\langle J_y\right\rangle^2).$$

Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle + \left\langle J_y^2\right\rangle).$$

- Simple algebra yields

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears instead of $\langle J_x\rangle^2$.

Normalized variables

- Let us introduce the normalized variables

$$\tilde{J}_m^{(n)} = \frac{J_m^{(n)}/j_n}{\mathcal{J}^{(n)}},$$

where $m = x, y$ and $n = \mathrm{a}, \mathrm{b}$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

and

$$\mathcal{J}^{(n)} = \left\langle \frac{(J_{\mathrm{x}}^{(n)})^2 + (J_{\mathrm{y}}^{(n)})^2}{j_n^2} \right\rangle^{\frac{1}{2}}.$$

- $\mathcal{J}^{(n)} \approx 1$ indicates a state close to be symmetric, general, $\mathcal{J}^{(n)} \leq 1$.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_{\mathrm{z}})^2 + \frac{1}{4}\right]\left[(\Delta \tilde{J}_{\mathrm{x}})^2 + (\Delta \tilde{J}_{\mathrm{y}})^2\right] \geq \frac{1}{4}.$$

We define

$$\begin{aligned} J_{\mathrm{z}}^{+} &= J_{\mathrm{z}}^{(a)} + J_{\mathrm{z}}^{(b)}, \\ \tilde{J}_{m}^{-} &= \tilde{J}_{m}^{(a)} - \tilde{J}_{m}^{(b)} \end{aligned}$$

for $m = \mathrm{x}, \mathrm{y}$.

The two-well entanglement criterion

Main result

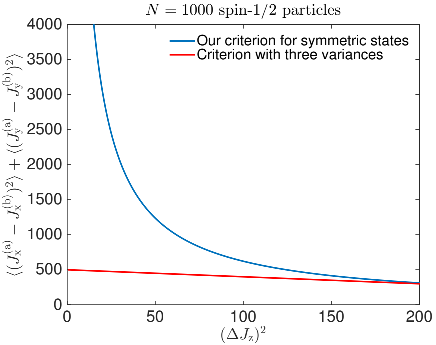
For separable states,

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[\langle(\tilde{J}_{\mathrm{x}}^{-})^2 + (\tilde{J}_{\mathrm{y}}^{-})^2\rangle\right] \geq f\left(\mathcal{J}^{(\mathrm{a})}, \mathcal{J}^{(\mathrm{b})}\right)$$

holds, where $f(x, y) = \frac{(x^2+y^2-1)^2}{xy}$,

Any state violating the inequality is entangled.

The two-well entanglement criterion II



- For symmetric states with $j_a = j_b = \frac{N}{4}$

$$\left[(\Delta J_z)^2 + \frac{1}{2}\right]\left[(\Delta(J_x^{(a)} - J_x^{(b)}))^2 + (\Delta(J_y^{(a)} - J_y^{(b)}))^2\right] \geq \frac{N^2}{16}.$$

Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Consider a density matrix

$$\varrho = \sum_{j_a, j_b} Q_{j_a, j_b} \varrho_{j_a, j_b},$$

  where ϱ_{j_a, j_b} are states with $2j_a$ and $2j_b$ particles in the two wells, Q_{j_a, j_b} are probabilities.
- ϱ is entangled iff at least one of the ϱ_{j_a, j_b} is entangled.
- Expectation value for operators depending on j_a and j_b

$$\langle A f(\hat{j}_a), \hat{j}_b) \rangle = \sum_{j_a, j_b} Q_{j_a, j_b} \langle A \rangle_{\varrho_{j_a, j_b}} f(j_a, j_b).$$

Correlations for Dicke states

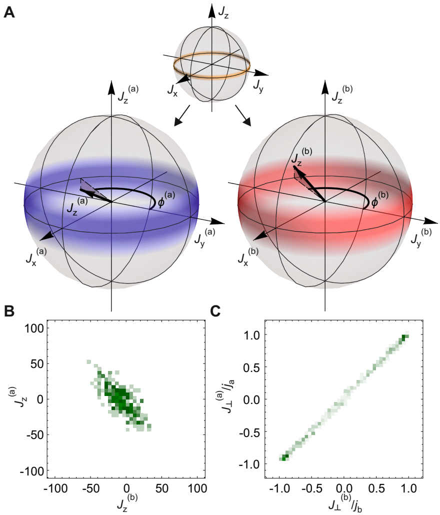
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_{\mathrm{x}}^{(\mathrm{a})} - J_{\mathrm{x}}^{(\mathrm{b})}))^2 &\approx 0, \\
(\Delta(J_{\mathrm{y}}^{(\mathrm{a})} - J_{\mathrm{y}}^{(\mathrm{b})}))^2 &\approx 0, \\
(\Delta J_{\mathrm{z}})^2 &= 0.
\end{aligned}
$$

- Measurements results on well "b" can be predicted from measurements on "a"

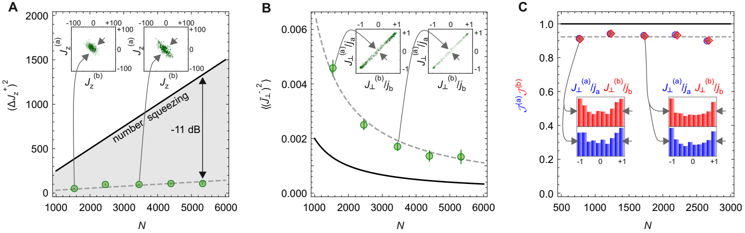
$$
\begin{aligned}
J_{\mathrm{x}}^{(\mathrm{b})} &\approx J_{\mathrm{x}}^{(\mathrm{a})}, \\
J_{\mathrm{y}}^{(\mathrm{b})} &\approx J_{\mathrm{y}}^{(\mathrm{a})}, \\
J_{\mathrm{z}}^{(\mathrm{b})} &= -J_{\mathrm{z}}^{(\mathrm{a})}.
\end{aligned}
$$

Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

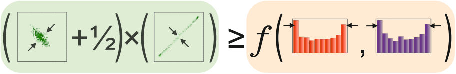
Further experimental results

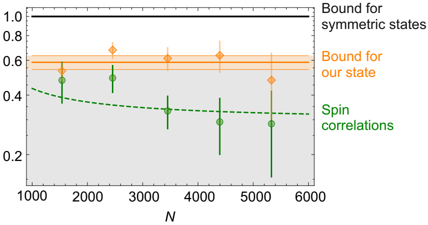


A. Black: shot-noise limit. Green circles: experiments.
B: Black: shot-noise limit. Green circles: experiments.
C: Black: perfect symmetry. Blue/red: values for the left/right well.

$$\mathcal{J}^{(n)} = \left\langle \frac{(J_{\mathrm{x}}^{(n)})^2 + (J_{\mathrm{y}}^{(n)})^2}{j_n^2} \right\rangle^{\frac{1}{2}} .$$

Violation of the criterion: entanglement is detected





For separable states,

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[\langle(\tilde{J}_{\mathrm{x}}^{-})^2 + (\tilde{J}_{\mathrm{y}}^{-})^2\rangle\right] \geq f\left(\mathcal{J}^{(\mathrm{a})}, \mathcal{J}^{(\mathrm{b})}\right)$$

holds, where $f(x,y) = \frac{(x^2+y^2-1)^2}{xy}$.

Collaborators on entanglement conditions for Dicke states

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	G. Tóth		C. Klempt

Summary

- Detection of bipartite entanglement close to Dicke states.
- Non-symmetric states and a varying particle number can also be handled.

K. Lange, J. Peise, B. Lücke, I. Kruse,
G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt,
Entanglement between two spatially separated atomic modes,
Science 360, 416 (2018).

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Appendix

Proof

Product states. For states of the form $|\Psi^{(\mathrm{a})}\rangle \otimes |\Psi^{(\mathrm{b})}\rangle$.

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[(\Delta \tilde{J}_{\mathrm{x}}^{-})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{-})^2\right]$$
$$= [(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4}) + (\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})] \cdot (\mathcal{V}^{(\mathrm{a})} + \mathcal{V}^{(\mathrm{b})})$$
$$\geq 4\sqrt{(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4})(\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})\mathcal{V}^{(\mathrm{a})}\mathcal{V}^{(\mathrm{b})}} \geq 1$$

holds, where we used the notation

$$\mathcal{U}^{(n)} = (\Delta J_{\mathrm{z}}^{(n)})^2, \qquad \mathcal{V}^{(n)} = (\Delta \tilde{J}_{\mathrm{x}}^{(n)})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{(n)})^2$$

for $n = \mathrm{a}, \mathrm{b}$. We used that

(i) $[\Delta(A^{(\mathrm{a})} + A^{(\mathrm{b})})]^2 = (\Delta A^{(\mathrm{a})})^2 + (\Delta A^{(\mathrm{b})})^2$,

(ii) Inequality between the arithmetic and the geometric mean,

(iii) Our number-phase like uncertainty.

Proof II

Using $\langle(\tilde{J}_x^{(n)})^2\rangle + \langle(\tilde{J}_y^{(n)})^2\rangle = 1$ for $n = \mathrm{a}, \mathrm{b}$, our inequality for product states yields

$$2\left[(\Delta J_z^+)^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq \mathcal{S},$$

where correlations between the two subsystems are characterized by

$$\mathcal{C} = \left\langle \frac{J_x^{(a)} J_x^{(b)} + J_y^{(a)} J_y^{(b)}}{j_a j_b} \right\rangle,$$

and

$$\mathcal{S} = \mathcal{J}^{(a)} \mathcal{J}^{(b)}.$$

$\mathcal{C}$ can be negative and $|\mathcal{C}| \leq \mathcal{S}$.

The normalization with the total spin will make it easier to adapt our criterion to experiments with a varying particle number in the ensembles.

Proof III

Separable states. We now consider a mixed separable state of the form $\varrho_{\rm sep} = \sum_k p_k |\Psi_k^{(\rm a)}\rangle \otimes |\Psi_k^{(\rm b)}\rangle$. For such states, we can write the following series of inequalities

$$2\left[(\Delta J_{\rm z}^{+})^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq 2\left[\sum_k p_k (\Delta J_{\rm z})^2{}_k + \frac{1}{2}\right]\left[\sum_k p_k\left(\mathcal{S}_k - \mathcal{C}_k\right)\right]$$

$$\geq 2\left[\sum_k p_k \sqrt{\left((\Delta J_{\rm z})^2{}_k + \frac{1}{2}\right)(\mathcal{S}_k - \mathcal{C}_k)}\right]^2 \geq \left(\sum_k p_k \sqrt{\mathcal{S}_k}\right)^2,$$

Subscript k refers to the $k^{\rm th}$ sub-ensemble $|\Psi_k^{(\rm a)}\rangle \otimes |\Psi_k^{(\rm b)}\rangle$.
(i) The first inequality in is due to $(\Delta J_{\rm z}^{+})^2$ and $\mathcal{S}$ being concave in the quantum state.
(ii) The second inequality is based on the Cauchy-Schwarz inequality.
(iii) The third inequality is the application of the previous inequality for all sub-ensembles.

Proof IV

Next, we find a lower bound on the RHS of the last inequality based on the knowledge of $\mathcal{J}^{(\mathrm{a})}$ and $\mathcal{J}^{(\mathrm{b})}$. We find that

$$\sum_k p_k \left(\mathcal{J}_k^{(\mathrm{a})}\mathcal{J}_k^{(\mathrm{b})}\right)^{1/2} \geq (\mathcal{J}^{(\mathrm{a})})^2 + (\mathcal{J}^{(\mathrm{b})})^2 - 1,$$

which is based on noting $(xy)^{1/4} \geq x + y - 1$ for $0 \leq x, y \leq 1$.

Using this to bound the RHS from below and dividing by $\mathcal{S}$ we obtain

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[2 - 2\frac{\mathcal{C}}{\mathcal{S}}\right] \geq \frac{\left[(\mathcal{J}^{(\mathrm{a})})^2 + (\mathcal{J}^{(\mathrm{b})})^2 - 1\right]^2}{\mathcal{S}}.$$