Witnessing metrologically useful multiparticle entanglement of Dicke states

G. Tóth[1,2,3] in collaboration with:

Theory: G. Vitagliano[1], I. Apellaniz[1],
M. Kleinmann[1], I.L. Egusquiza[1], O. Gühne[4]

Cold gas exp.: B. Lücke[5], J. Peise[5], J. Arlt[5], L. Santos[5], C. Klempt[5]

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary
[4]University of Siegen, Germany
[5]Leibniz Universität Hannover, Germany

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Why multipartite entanglement is important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

- Separable states remain separable under local operations. ("Local operations and classical communication")
- Separable states can be cerated without real quantum interaction. They are the "boring" states.

k-producibility/k-entanglement

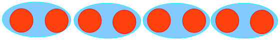
A pure state is *k*-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

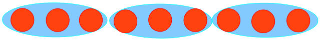
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.



two-producible



three-producible

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

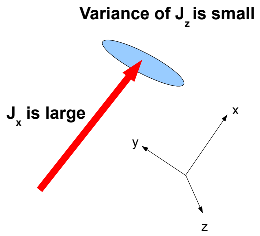
The standard spin-squeezing criterion

The spin squeezing criteria for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

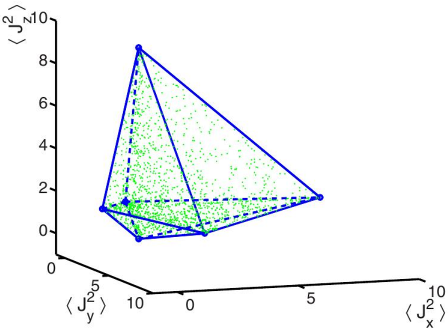
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke state)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]
[Singlets: Behbood *et al.*, Phys. Rev. Lett. 2014; GT, Mitchell, New. J. Phys. 2010.]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{\rm 2p}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{\rm 2p}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{\rm 2p} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{\rm 2p}$.
- Still, as we will see, we can even detect multipartite entanglement!

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011; GT, PRA 2012; GT and Apellaniz, JPHYSA, 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$
$$\langle J_z^2 \rangle = 0.$$

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001);
experimental test: C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler,

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

  and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

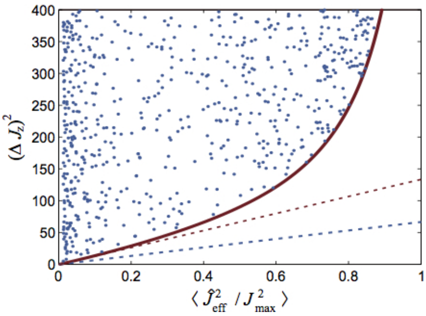
which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Concrete example

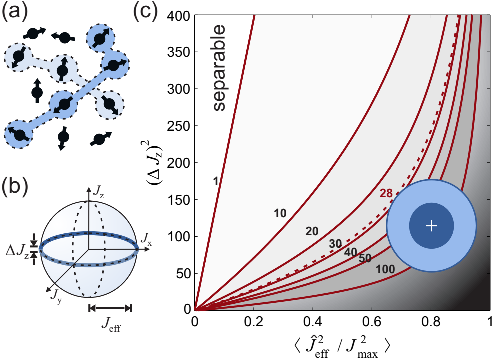


- $N = 8000$ particles, and $J_{\text{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in
  [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- Red dashed line: tangent of our curve.

Outline

Experimental results

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



Giuseppe Vitagliano

[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014)]

Our recent work: General criterion for larger spins

- Dicke states of particles with a spin larger than 1/2 have been created.
- See, for example, T. M. Hoang, M. Anquez, M. J. Boguslawski, H. M. Bharath, B. A. Robbins, and M. S. Chapman, arXiv:1512.06766 (2015).

[Vitagliano *et al.*, arXiv:1605.07202.]

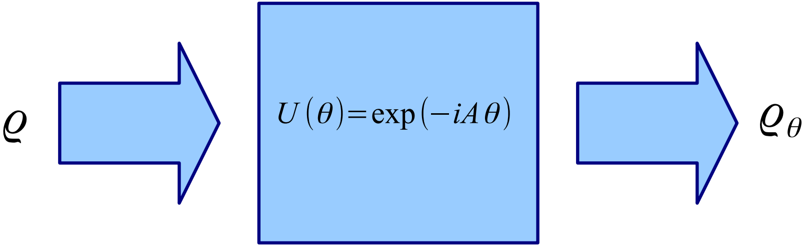
- Note also alternative methods for entanglement detection for large spins.

[G. Vitagliano *et al.*, Phys. Rev. A 89, 032307 (2014); G. Vitagliano *et al.*, Phys. Rev. Lett. 107, 240502 (2011).]

Outline

Quantum metrology

- Fundamental task in metrology



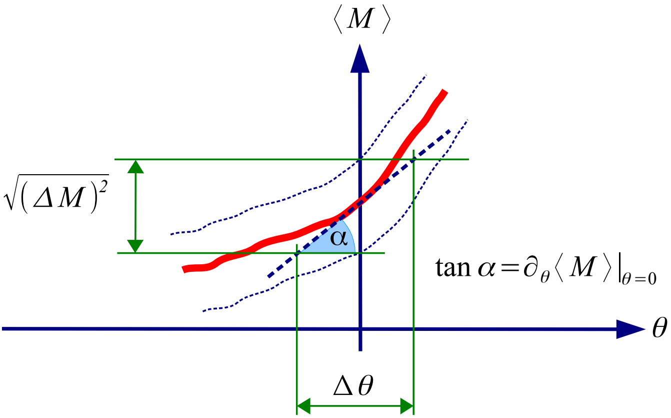
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Outline

Metrology with Dicke states

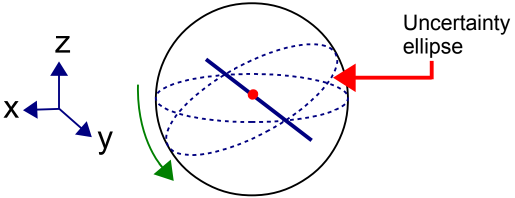
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)

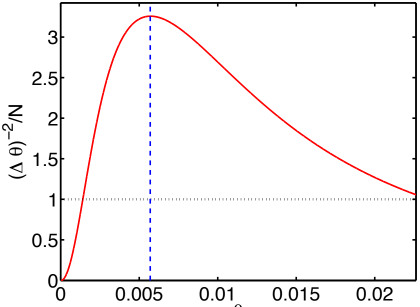


Metrology with Dicke states II

We measure $\langle J_z^2 \rangle$ to estimate θ. The precision is given by the error-propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta J_z^2)^2}{|\partial_\theta \langle J_z^2 \rangle|^2}.$$

- Precision as a function of θ for some noisy Dicke state



Formula for maximal precision

Parameter value for the maximum

$$\tan^2 \theta_{\text{opt}} = \sqrt{\frac{(\Delta J_z^2)^2}{(\Delta J_x^2)^2}}.$$

Consistency check: for the noiseless Dicke state we have $(\Delta J_z^2)^2 = 0$, hence $\theta_{\text{opt}} = 0$.

Iagoba Apellaniz

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision II

Maximal precision with a closed formula

$$(\Delta\theta)^2_{\rm opt} = \frac{2\sqrt{(\Delta J_z^2)^2(\Delta J_x^2)^2} + 4\langle J_x^2\rangle - 3\langle J_y^2\rangle - 2\langle J_z^2\rangle(1+\langle J_x^2\rangle) + 6\langle J_z J_x^2 J_z\rangle}{4(\langle J_x^2\rangle - \langle J_z^2\rangle)^2}.$$

- Given in terms of collective observables, like spin-squeezing criteria.
- Metrological usefulness can be verified without carrying out the metrological task.

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, New J. Phys. 17, 083027 (2015).]

Experimental test of our formula

- Trying the bound for the experimental data for $N = 7900$ particles

$$\langle J_z^2 \rangle = 112 \pm 31, \qquad \langle J_z^4 \rangle = 40 \times 10^3 \pm 22 \times 10^3,$$
$$\langle J_x^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6, \quad \langle J_x^4 \rangle = 6.2 \times 10^{13} \pm 0.8 \times 10^{13}.$$

- Hence, we obtain

$$\frac{(\Delta\theta)^{-2}_{\mathrm{opt}}}{N} \geq 3.7 \pm 1.5.$$

- Remember, for states for at most k-particle entanglement we have

$$(\Delta\theta)^{-2} \leq F_Q[\varrho, J_l] \leq kN.$$

- Thus, four-particle entanglement is detected for this particular measurement.

Comparison with the quantum Fisher information

- For the noiseless Dicke state, the optimal operator to measure is

$$M = J_z^2.$$

- For a noisy Dicke state, this is not true any more. In this case it can happen that

$$(\Delta\theta)^{-2} = \frac{|\partial_\theta \langle J_z^2 \rangle|^2}{(\Delta J_z^2)^2} \ll F_Q[\varrho, J_y].$$

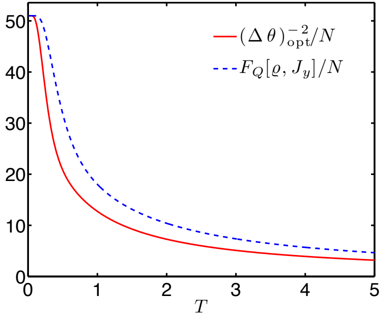
- We should estimate the quantum Fisher information.

Comparison with the quantum Fisher information II

- Noisy states

$$\varrho_{\rm th}(T) \propto \sum_{m=0}^{N} e^{-\frac{(m-N/2)^2}{T}} |{\rm D}_N^{(m)}\rangle\langle{\rm D}_N^{(m)}|,$$

- Here $T = 0$ perfect symmetric Dicke state, $T > 0$ noisy state. $N = 100$ particles



Outline

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Legendre transform

- Optimal linear lower bound on a convex function $g(\varrho)$ based on an operator expectation value $w = \langle W \rangle_\varrho = \mathrm{Tr}(W\varrho)$

$$g(\varrho) \geq rw - const.,$$

where $w = \mathrm{Tr}(\varrho W)$.

- For every r there is a "*const.*" that makes the relation an optimal linear lower bound.

- How large is "*const.*"? It can be obtained as

$$g(\varrho) \geq \mathcal{B}(w) := rw - \hat{g}(rW),$$

where $\hat{g}$ is the Legendre transform

$$\hat{g}(W) = \sup_\varrho [\langle W \rangle_\varrho - g(\varrho)].$$

[O. Gühne, M. Reimpell, and R. F. Werner, PRL 98, 110502 (2007);
J. Eisert, F. G. S. L. Brandao, and K. M. R. Audenaert, NJP 9, 46 (2007).]

Legendre transform II

- Tight lower bound can be obtained if we optimize over r as

$$g(\varrho) \geq \mathcal{B}(w) := \sup_r \left[rw - \hat{g}\left(rW\right)\right],$$

where again $w = \mathrm{Tr}(\varrho W)$.

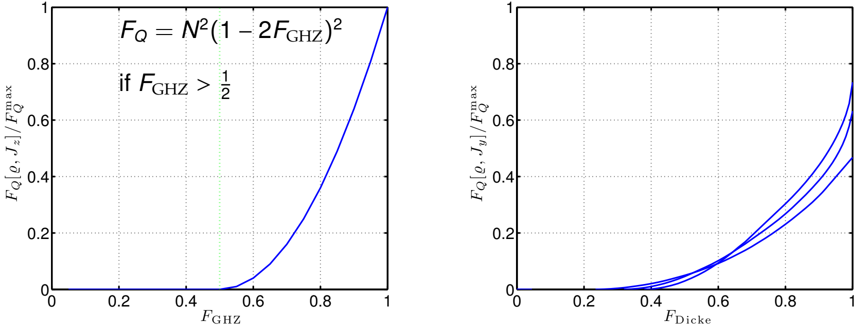
- The quantum Fisher information is given as a convex roof. For this case, it is sufficient to carry out an optimization over pure states

$$\hat{g}(W) = \sup_\Psi [\langle W \rangle_\Psi - g(\Psi)].$$

- With further simplifications, an optimization over a single real variable is needed.

Witnessing the quantum Fisher information based on the fidelity

- Let us bound the quantum Fisher information based on some measurements. First, consider small systems.
  [See also Augusiak *et al.*, 1506.08837.]



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, arXiv:1511.05203.]

Bounding the qFi based on collective measurements

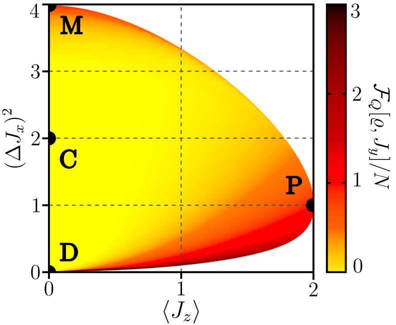
Bound for the quantum Fisher information for spin squeezed states (Pezze-Smerzi bound)

$$F[\varrho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Bounding the qFi based on collective measurements II

- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles

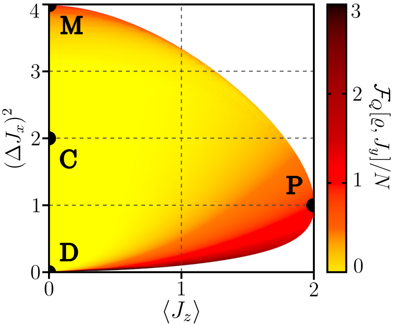


P=fully polarized state, D=Dicke state, C=completely mixed state, M=mixture of $|00..000\rangle_x$ and $|11..111\rangle_x$

[Apellaniz, Kleinman, Gühne, GT, arXiv:1511.05203.]

Bounding the qFi based on collective measurements III

- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles

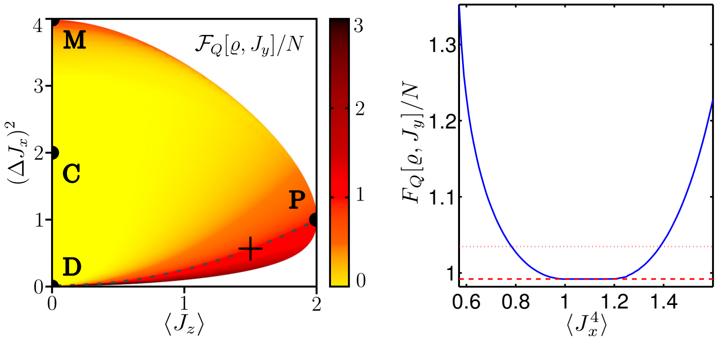


On the bottom part of the figure ($(\Delta J_x)^2 < 1$) the bound is very close to the Pezze-Smerzi bound!

[Apellaniz, Kleinman, Gühne, GT, arXiv:1511.05203.]

Bounding the qFi based on collective measurements IV

- The bound can be obtained if additional expectation value, i.e., $\langle J_x^2 \rangle$ is measured, or we assume symmetry:



[Apellaniz, Kleinman, Gühne, GT, arXiv:1511.05203.]

Spin squeezing experiment

- Experiment with $N = 2300$ atoms,

$$\xi_s^2 = -8.2\text{dB} = 10^{-8.2/10} = 0.1514.$$

[C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, Nature 464, 1165 (2010).]

- We choose

$$\langle J_z \rangle = \alpha \frac{N}{2},$$

with $\alpha = 0.85$. (Almost fully polarized.)

- The Pezze-Smerzi bound is:

$$\frac{\mathcal{F}_Q[\varrho_N, J_y]}{N} \geq \frac{1}{\xi_s^2} = 6.605,$$

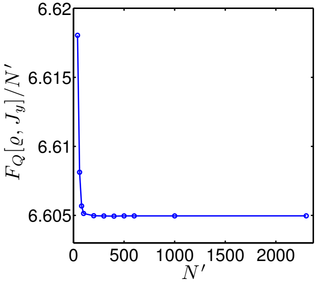
[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Spin squeezing experiment

- Solution starting with small systems, using

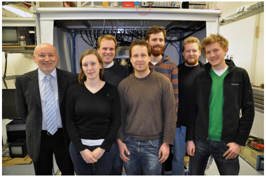
$$\langle J_z \rangle = \frac{N'}{2}\alpha,$$
$$(\Delta J_x)^2 = \xi_s^2 \frac{N'}{4}\alpha^2.$$

- We get 6.605!!



- Proof that the formula is optimal, and that our method works.

Project participants



C. Klempt, B. Lücke, J. Mahnke
W. Ertmer, I. Geisel, J. Peise, S. Coleman

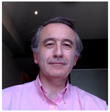
L. Santos

Hannover

G. Vitagliano

I. Apellaniz

M. Kleinmann



I.L. Egusquiza

Bilbao (G.T.)

O. Gühne

Siegen

Summary

- Detection of multipartite entanglement and metrological usefulness close to Dicke states, by measuring collective quantities only.

Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt,
PRL 112, 155304 (2014);
Vitagliano, Apellaniz, Kleinmann, Lücke, Klempt, GT,
arXiv:1605:07202;
Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);
Apellaniz, Kleinmann, Gühne, Tóth, arxiv: arXiv:1511.05203.

THANK YOU FOR YOUR ATTENTION!

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