

Quantum Wasserstein distance based on an optimization over separable states

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1 Motivation

- Connecting Wasserstein distance to entanglement theory?

2 Background

- Quantum Wasserstein distance
- Quantum Fisher information

3 Wasserstein distance and separable states

- Quantum Wasserstein distance based on an optimization over separable states
- Relation to entanglement conditions

Motivation

- Many distance measures are maximal for orthogonal states.
- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.
- For the quantum case, surprisingly, the self-distance can be nonzero.
- Can we connect these to entanglement theory?

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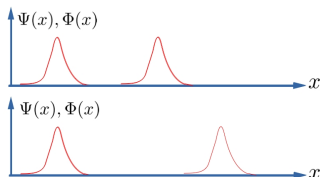
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An important property of the Wasserstein distance

- Many distance measures are maximal for orthogonal states, e.g., for the following state-pairs.



- In the second example, the two states are further apart from each other, based on common sense.
- Wasserstein distance can recognize this since it is related to the "cost of moving sand from a distribution to the other one."
- It can be used for machine learning.

Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices ϱ and σ is

$$D_{\text{DPT}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^N \text{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],$$

s. t.

$$\begin{aligned} \varrho_{12} &\in \mathcal{D}, \\ \text{Tr}_2(\varrho_{12}) &= \varrho^T, \\ \text{Tr}_1(\varrho_{12}) &= \sigma, \end{aligned}$$

where \mathcal{D} is the set of density matrices, and H_n are Hermitian matrices.

- Note the relation to the representability problem.

G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, *Ann. Henri Poincaré* 22, 3199 (2021).

Self-distance can be nonzero (unlike in the classical case)

- The self-distance of a state is

$$D_{\text{DPT}}(\varrho, \varrho)^2 = \sum_{n=1}^N I_{\varrho}(H_n),$$

where the Wigner-Yanase skew information is defined as

$$I_{\varrho}(H) = \text{Tr}(H^2\varrho) - \text{Tr}(H\sqrt{\varrho}H\sqrt{\varrho}).$$

- This connects connects Wasserstein distance and quantum metrology.

G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, *Ann. Henri Poincaré* 22, 3199 (2021).

Wasserstein distance between a pure state

$\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ

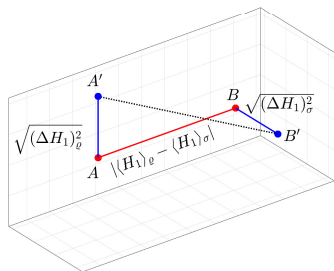
- The distance is given as

$$\begin{aligned} D_{\text{DPT}}(\varrho, \sigma)^2 &= \frac{1}{2} \sum_{n=1}^N [(\Delta H_n)^2_{\varrho} + (\Delta H_n)^2_{\sigma} + (\langle H_n \rangle_{\varrho} - \langle H_n \rangle_{\sigma})^2], \end{aligned}$$

see the following figure, where $(\Delta H_n)^2$ is the variance.

Wasserstein distance between a pure state

$\rho = |\Psi\rangle\langle\Psi|$ and a mixed state σ II



- $N = 1$ with operator H_1 .
- The quantum Wasserstein distance equals $1/\sqrt{2}$ times the usual Euclidean distance between A' and B' .

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

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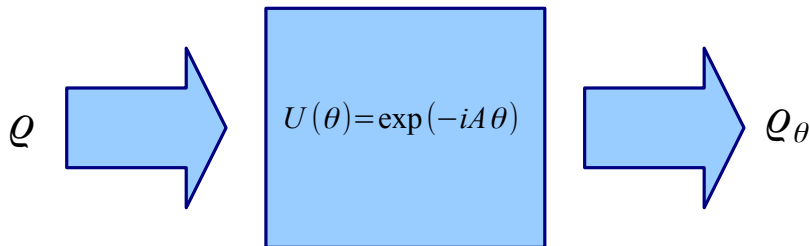
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Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the **quantum Fisher information**, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance times four

$$F_Q[\varrho, A] = 4 \min_{\{\rho_k, |\psi_k\rangle\}} \sum_k \rho_k (\Delta A)_{\psi_k}^2,$$

where

$$\varrho = \sum_k \rho_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

Formula based on concave roofs

The variance is the concave roof of itself

$$(\Delta A)^2_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

A single relation for the QFI and the variance

For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)_{\psi_k}^2 \leq (\Delta A)_{\varrho}^2,$$

where the upper and the lower bounds are both **tight**.

- Note that

$$\frac{1}{4}F_Q[\varrho, A] \leq (\Delta A)_{\varrho}^2,$$

where for pure states we have an equality.

- The QFI is strongly related to the variance.

Formula based on an optimization in the two-copy space

- Two-copy formulation for the variance

$$(\Delta H)^2_{\Psi} = \text{Tr}(\Omega|\Psi\rangle\langle\Psi| \otimes |\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\begin{aligned} F_Q[\varrho, H] &= \min_{\varrho_{12}} && 4\text{Tr}(\Omega\varrho_{12}), \\ &\text{s. t.} && \varrho_{12} \in \mathcal{S}', \\ &&& \text{Tr}_2(\varrho_{12}) = \varrho. \end{aligned}$$

Here \mathcal{S}' is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne, Evaluating convex roof entanglement measures, Phys. Rev. Lett. 114, 160501 (2015); GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

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Quantum Wasserstein distance based on an optimization over separable states

- **Definition**—We can also define

$$D_{\text{DPT,sep}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^N \text{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],$$

s. t.

$$\begin{aligned} \varrho_{12} &\in \mathcal{S}, \\ \text{Tr}_2(\varrho_{12}) &= \varrho^T, \\ \text{Tr}_1(\varrho_{12}) &= \sigma, \end{aligned}$$

where \mathcal{S} is the set of separable states.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

Quantum Wasserstein distance based on an optimization over separable states II

- For two-qubits, **it is computable numerically** with semidefinite programming.
- For systems of larger dimensions, one can obtain a **very good lower bound** based on an optimization over states with a positive partial transpose (PPT).
- Even better lower bounds can be obtained.

P. Horodecki, Phys. Lett. A 232, 333 (1997);

A. Peres, Phys. Rev. Lett. 77, 1413 (1996);

A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 69, 022308 (2004).

Self-distance

- The self-distance for $N = 1$ is

$$D_{\text{DPT,sep}}(\varrho, \varrho)^2 = \frac{1}{4} F_Q[\varrho, H_1].$$

- Note that

$$I_\varrho(A) \leq \frac{1}{4} F_Q[\varrho, A] \leq (\Delta A)_\varrho^2.$$

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Entanglement of ϱ_{12}

- In general,

$$D_{\text{DPT,sep}}(\varrho, \sigma) \geq D_{\text{DPT}}(\varrho, \sigma).$$

- If the relation

$$D_{\text{DPT,sep}}(\varrho, \sigma) > D_{\text{DPT}}(\varrho, \sigma)$$

holds, then all the optimal ϱ_{12} couplings for $D_{\text{DPT}}(\varrho, \sigma)$ are entangled.

- Allowing an entangled ϱ_{12} decreases the cost!
- Thus, **an entangled ϱ_{12} can be cheaper than a separable one.**

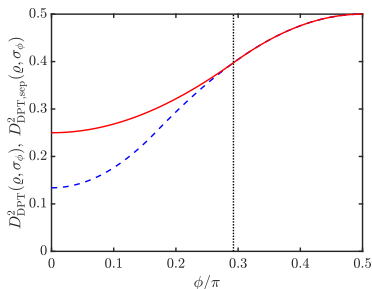
Comparison of the two types of Wasserstein distance

- Let us consider the distance between two single-qubit mixed states

$$\rho = \frac{1}{2}|1\rangle\langle 1|_x + \frac{1}{2} \cdot \frac{\mathbb{1}}{2},$$

and

$$\sigma_\phi = e^{-i\frac{\sigma_y}{2}\phi} \rho e^{i\frac{\sigma_y}{2}\phi}.$$



Bounds on the distance

- **Entanglement condition:** Let us choose a set of H_n such that

$$\frac{1}{2} \sum_n \langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \rangle \geq \text{const.}$$

holds for separable states.

- E. g., $\{H_n\} = \{j_x, j_y, j_z\}$ and "const." = j .

- If the inequality

$$D_{\text{DPT}}(\rho, \sigma) < \text{const.}$$

holds, then all optimal ρ_{12} states for $D_{\text{DPT}}(\rho, \sigma)$ are entangled.

- Then, we will have a **minimal distance**

$$D_{\text{DPT,sep}}(\rho, \sigma) \geq \text{const.}$$

Summary

- For the quantum Wasserstein distance, we restrict the optimization to separable states.
- Then, the self-distance equals the quantum Fisher information over four.
- We found a fundamental connection from quantum optimal transport to quantum entanglement theory and quantum metrology.

G. Tóth and J. Pitrik, [arXiv:2209.09925](https://arxiv.org/abs/2209.09925).

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