# Evaluation of convex roof entanglement measures

G. Tóth[1,2,3], T. Moroder[4], and O. Gühne[4]

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary
[4]University of Siegen, Germany

DPG Heidelberg, 25 March 2015


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
ICELAND
Reykjavík
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangel
Luleå
Oulu
Umeå
Lake Onega
Trondheim
Gulf of Bothnia
Tampere
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
Bergen
Gävle
Turku
Helsinki
St. Petersburg
ORKNEY ISLANDS
Oslo
ALAND ISLANDS
Gulf of Finland
Rockall (U.K.)
Stockholm
Tallinn
ESTONIA
HEBRIDES
Stavanger
North Atlantic Ocean
Aberdeen
Göteborg
Gotland
LATVIA
Riga
Glasgow
Edinburgh
North Sea
Baltic Sea
Öland
DENMARK
Copenhagen
Malmö
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
UNITED KINGDOM
Belfast
Kaliningrad
RUSSIA
Minsk
Mahilyow
BELARUS
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Leeds
Liverpool
Manchester
Bornholm
Gdańsk
Hrodna
Homyel
Hamburg
Bremen
Berlin
Poznań
POLAND
Warsaw
Brest
Birmingham
Amsterdam
NETH.
Łódź
Kiev
Cardiff
London
Rotterdam
Essen
Cologne
Leipzig
Wrocław
Rivne
Celtic Sea
English Channel
Brussels
BEL.
Lille
Bonn
GERMANY
Prague
Kraków
UKRAINE
L'viv
Guernsey (U.K.)
Jersey (U.K.)
Frankfurt am Main
CZECH REPUBLIC
Brno
Chernivtsi
Mykolayiv
LUX.
Luxembourg
SLOVAKIA
Chisinau
Paris
Stuttgart
Bratislava
Iaşi
MOLDOVA
Strasbourg
Munich
Vienna
Budapest
Cluj-Napoca
Nantes
AUSTRIA
LIECH.
HUNGARY
ROMANIA
Bay of Biscay
FRANCE
Zürich
Bern
SWITZ.
Geneva
Vaduz
Ljubljana
SLOVENIA
Zagreb
Bucharest
Constanța
A Coruña
Lyon
Milan
Venice
Turin
Bordeaux
Bilbao
CROATIA
BOSNIA AND HERZEGOVINA
Belgrade
SERBIA AND MONTENEGRO
Varna
Genoa
Toulouse
MONACO
SAN MARINO
Sarajevo
BULGARIA
Sofia
Porto
Andorra la Vella
Marseille
ANDORRA
Ligurian Sea
Florence
Adriatic Sea
Skopje
F.Y.R.O.M.
Istanbul
Zaragoza
Corsica
ITALY
Rome
PORTUGAL
Madrid
Barcelona
VATICAN CITY
Tirana
ALB.
Thessaloníki
Lisbon
Balearic Sea
Naples
TURKEY
SPAIN
Valencia
Tyrrhenian Sea
GREECE
Aegean Sea
Sardinia
Athens
Sevilla
BALEARIC ISLANDS
Cagliari
Ionian Sea
Mediterranean Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Algiers
Tunis
Crete
Melilla (SPAIN)
Oran
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

# Outline

1. **Motivation**
   - Why entanglement quantification is important?

2. **Calculating entanglement measures**
   - Convex roof of the entropy
   - Tangle
   - Other quantities
   - Even tighter lower bounds

# Why entanglement quantification is important?

- Many experiments are aiming to create entangled states.
- We need to calculate entanglement measures for these states.
- Apart from trivial system sizes, we cannot do it.

# Outline

1. **Motivation**
   - Why entanglement quantification is important?

2. **Calculating entanglement measures**
   - Convex roof of the entropy
   - Tangle
   - Other quantities
   - Even tighter lower bounds

# Entanglement

## The entanglement of a bipartite quantum state

For pure states, it is defined as

$$E(|\Psi\rangle) = S[\mathrm{Tr}_1(|\Psi\rangle)],$$

for pure states, where $S$ is an entropy.

For mixed states, it is defined with a convex roof as

$$E(\varrho) = \min_{\{p_k,|\Psi_k\rangle\}} \left( \sum_k p_k E(|\Psi_k\rangle) \right),$$

where $\{p_k, |\Psi_k\rangle\}$ is a decomposition to pure states

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

# Linear entropy for pure states

- Linear entropy
$$S_{\mathrm{lin}}(\varrho) = 1 - \mathrm{Tr}(\varrho^2).$$

- Known: linear entropy of entanglement for pure states can be defined as an expectation value on two copies ($AB$ and $A'B'$) as
$$E_{\mathrm{lin}}(|\Psi\rangle) = \mathrm{Tr}[\mathcal{A}_{\mathrm{AA'}} \otimes \mathbb{1}_{BB'}(|\Psi\rangle\langle\Psi|)_{AB} \otimes (|\Psi\rangle\langle\Psi|)_{A'B'}],$$
where
$$\mathcal{A}_{\mathrm{AA'}} := (\mathbb{1} - \mathcal{F})_{\mathrm{AA'}}$$
and $\mathcal{F}$ is the flip operator.

# Linear entropy for mixed states: convex roof

- For mixed states

$$\begin{aligned} E_{\mathrm{lin}}(\varrho) &= \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k E_{\mathrm{lin}}(|\Psi_k\rangle) = \\ &= \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k \mathrm{Tr}(\mathcal{A}_{AA'}|\Psi_k\rangle\langle\Psi_k|^{\otimes 2}) \\ &= \min_{\omega_{12}} \mathrm{Tr}(\mathcal{A}_{AA'}\omega_{12}), \end{aligned}$$

where $\omega_{12}$ are symmetric separable states, i.e.,

$$\omega_{12} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \otimes |\Psi_k\rangle\langle\Psi_k|.$$

- **This is the key step in our approach.**

# Surprise 1

- Mapping of the problem

  Optimization over decompositions $\longrightarrow$ Optimization over symmetric separable states

- We connected the separability theory to a general mathematical problem.

# How to calculate it

- The convex roof of the linear entropy can be written as

$$\begin{aligned} E_{\text{lin}}(\varrho) = \min_{\omega_{12}} \quad & \mathrm{Tr}(\mathcal{A}_{AA'}\omega_{12}), \\ \text{s.t.} \quad & \omega_{12} \text{ is symmetric, separable,} \\ & \omega_1 = \varrho, \end{aligned}$$

  where $\omega_1 \equiv \mathrm{Tr}_2(\omega_{12})$.

- A lower bound can be obtained as with the PPT condition

$$\begin{aligned} E_{\text{lin}}(\varrho) = \min_{\omega_{12}} \quad & \mathrm{Tr}(\mathcal{A}_{AA'}\omega_{12}), \\ \text{s.t.} \quad & \omega_{12} \text{ is symmetric PPT,} \\ & \omega_1 = \varrho, \end{aligned}$$

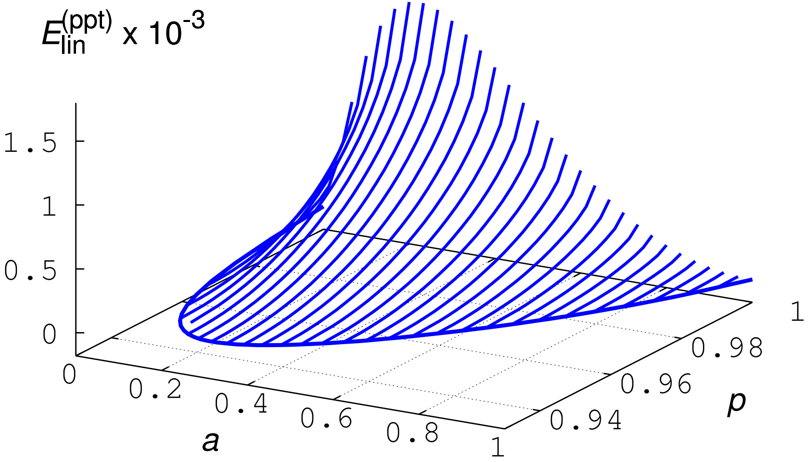
  where $\omega_1 \equiv \mathrm{Tr}_2(\omega_{12})$. This is a semidefinite program.

# Surprise 2

- The lower bound
  - is nonzero for all states with a non-positive semidefinite partial transpose (NPPT).
  - is nonzero for some states with a positive semidefinite partial transpose (PPT).
- For all non-PPT states and for all states that do not have a $2:2$ symmetric extension we have a nonzero bound.
- Moreover, for all states having a 2:2 PPT symmetric extension the bound is zero.
  [Extensions: Doherty, Parrilo, Spedalieri, PRA 69, 022308 (2004)]

# Example: Entanglement of a PPT state

- $3 \times 3$ Horodecki state mixed with white noise.
- $a$ =parameter of the state, $1 - p$ =noise fraction



# Outline

# Wootters' Tangle

- The well-known tangle for three-qubits can be defined as a fourth-order polynomial in expectation values.
  [A. Osterloh and J. Siewert, Phys. Rev. A 86, 042302 (2012).]

- Hence, it can be obtained as an optimization over four-partite symmetric separable states

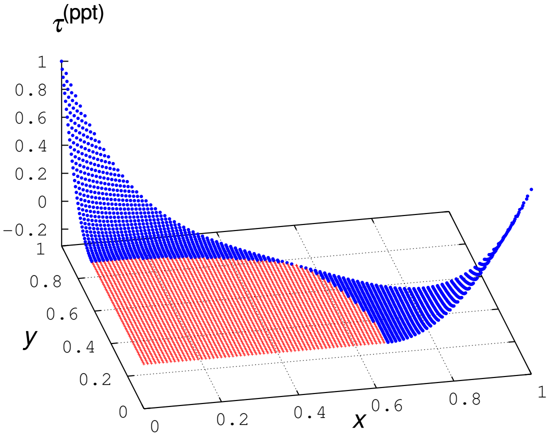
$$
\begin{aligned}
\tau(\varrho) = \min_{\omega_{1234}} \quad & \mathrm{Tr}(T\omega_{1234}), \\
\text{s.t.} \quad & \omega_{1234} \text{ symmetric, fully separable}, \\
& \omega_1 = \varrho,
\end{aligned}
$$

  where $T$ is an operator (4 parties with 3 qubits each).

- Similar idea works: replace separable states by PPT states.

# Example: tangle of a two-parameter family of states

$$\varrho(x,y) = x|\mathrm{GHZ}^+\rangle\langle\mathrm{GHZ}^+| + y|\mathrm{GHZ}^-\rangle\langle\mathrm{GHZ}^-| + (1-x-y)|\mathrm{W}\rangle\langle\mathrm{W}|$$



# Outline

1. Motivation
   - Why entanglement quantification is important?

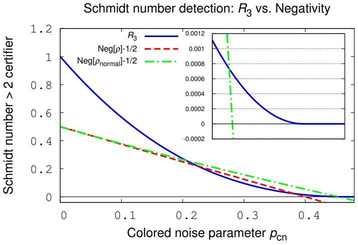
2. **Calculating entanglement measures**
   - Convex roof of the entropy
   - Tangle
   - Other quantities
   - Even tighter lower bounds

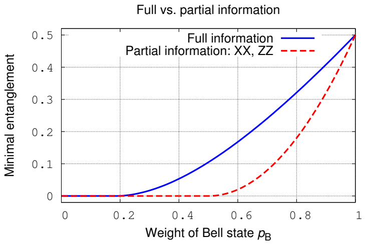
# Other quantities

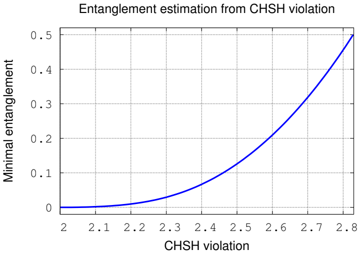
- Schmidt number. I.e., the convex roof of $R_3(|\Psi\rangle) = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k$ tells us whether the Schmidt number is larger than 2.
- Entanglement vs. CHSH violation
- Lower bound on entanglement based on some measurement results
- Concave roof instead of convex roofs: E. of assistance
- Lower bound on quantum Fisher information based some measurement results.
  [Tóth, Petz, PRA 2013.]
- One can get even a witness!!

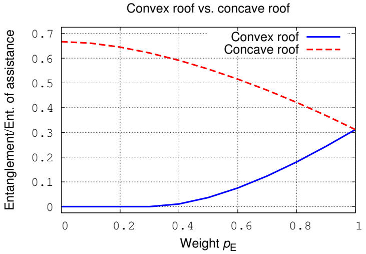
[For references, please see our work on the arxive.]

# Examples


Schmidt number detection: $R_3$ vs. Negativity
Schmidt number > 2 certifier
Colored noise parameter $p_{cn}$
$R_3$
Neg[$\rho$]-1/2
Neg[$\rho_{normal}$]-1/2
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
-0.0002


Full vs. partial information
Minimal entanglement
Weight of Bell state $p_B$
Full information
Partial information: XX, ZZ
0.5
0.4
0.3
0.2
0.1
0
0.6
0.8
1


Entanglement estimation from CHSH violation
Minimal entanglement
CHSH violation
0.5
0.4
0.3
0.2
0.1
0
2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
2.8


Convex roof vs. concave roof
Entanglement/Ent. of assistance
Weight $p_E$
Convex roof
Concave roof
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.8
1

# Outline

1. **Motivation**
   - Why entanglement quantification is important?

2. **Calculating entanglement measures**
   - Convex roof of the entropy
   - Tangle
   - Other quantities
   - Even tighter lower bounds

# A series of tighter and tighter lower bounds

- To strengthen the bound, a criterion stronger than PPT must be employed.
- For example, the method of PPT symmetric extensions can be used.
  [Doherty, Parrilo, Spedalieri, Phys. Rev. A 69, 022308 (2004)]
- Sequence of lower bounds $E_{\mathrm{lin}}^{(n)}$ with increasing accuracies.
- Calculation: semidefinite program.

# Summary

- We showed how to obtain a good lower bound on quantities defined with convex roofs.
- We used it for calculating entanglement measures and the tangle, and several other quantities.

See:
GT, T. Moroder, and O. Gühne,
Evaluation of convex roof entanglement measures,
Phys. Rev. Lett, in press; arxiv:1409.3806.

www.gtoth.eu

THANK YOU FOR YOUR ATTENTION!

erc European Research Council

MINISTERIO DE CIENCIA E INNOVACIÓN

EUSKO JAURLARITZA GOBIERNO VASCO