Quantum states with a positive partial transpose are useful for metrology

Géza Tóth[1,2,3] and Tamás Vértesi[4] 

[1]Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary
[4]Institute for Nuclear Research, Hungarian Academy of Sciences, Debrecen, Hungary

DPG, Erlangen
9 March 2018

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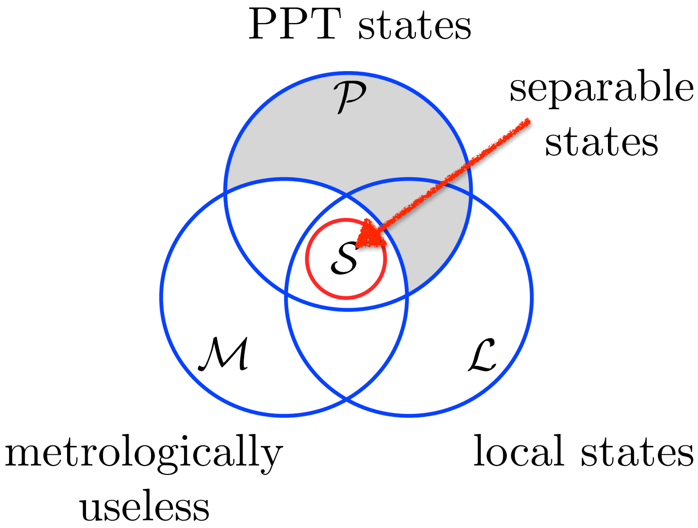
What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

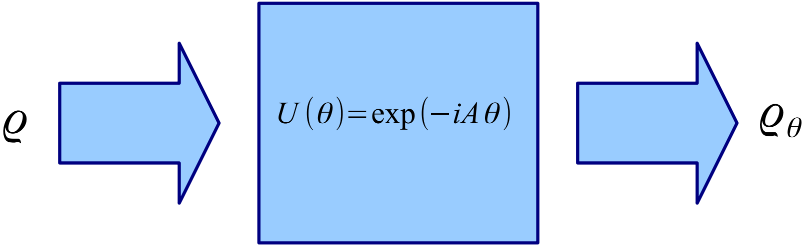
What are entangled states useful for?



Outline

Quantum metrology

- Fundamental task in metrology



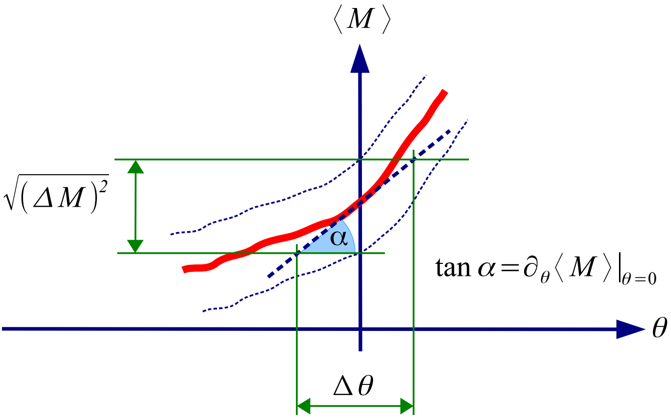
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- Shot-noise limit: For separable states

$$F_Q[\varrho, J_l] \le N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- A quantum state is "useful" if it violates the above inequality.

- Heisenberg limit: For entangled states

$$F_Q[\varrho, J_l] \le N^2, \qquad l = x, y, z.$$

where the bound can be saturated.

Outline

Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.

  [P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012).]

- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.

  [Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015).]

on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

Outline

Our results

We look for bipartite PPT entangled states and multipartite states that are PPT with respect to all partitions.

Maximizing the QFI for PPT states: brute force

- Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.

- Not guaranteed to find the global maximum.

- Note: Finding the *minimum* is possible!

Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2} = \frac{\langle i[M, A]\rangle^2}{(\Delta M)^2} \quad \text{(dynamics is } U = e^{-iA\theta}\text{)}.$$

- The bound is sharp

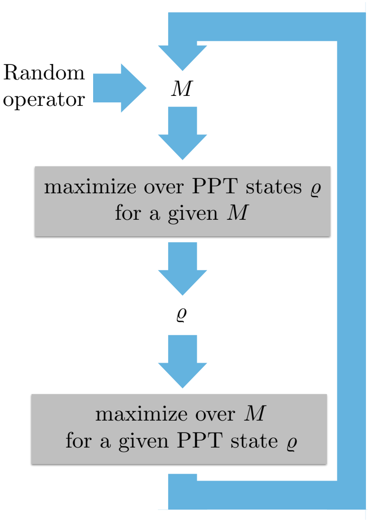
$$F_Q[\varrho, A] = \max_M \frac{\langle i[M, A]\rangle_\varrho^2}{(\Delta M)^2}.$$

[M. G. Paris, Int. J. Quantum Inform. 2009. Used, e.g., in F. Fröwis, R. Schmied, and N. Gisin, 2015; I. Appelaniz *et al.*, NJP 2015.]

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M, A]\rangle_\varrho^2}{(\Delta M)^2}.$$

Sew-saw algorithm for maximizing the precision



See also K. Macieszczak, arXiv:1312.1356v1 for an iterative algorithm for optimizing over noisy states.

Maximize over PPT states for a given M

Best precision for PPT states for a given operator M can be obtained by a semidefinite program.

Proof.—Let us define first

$$\begin{aligned} f_M(X,Y) = \min_{\varrho} \quad & \mathrm{Tr}(M^2\varrho), \\ \text{s.t.} \quad & \varrho \geq 0, \varrho^{\mathrm{T}k} \geq 0 \text{ for all } k, \mathrm{Tr}(\varrho) = 1, \\ & \langle i[M,A] \rangle = X \text{ and } \langle M \rangle = Y. \end{aligned}$$

The best precsion for a given M and for PPT states is

$$(\Delta\theta)^2 = \min_{X,Y} \frac{f_M(X,Y) - Y^2}{X^2}.$$

The state giving the best precision is $\varrho_{\mathrm{PPTopt}}$.

Maximize over M for a given PPT state

For a state ϱ, the best precision is obtained with the operator given by the symmetric logarithmic derivative

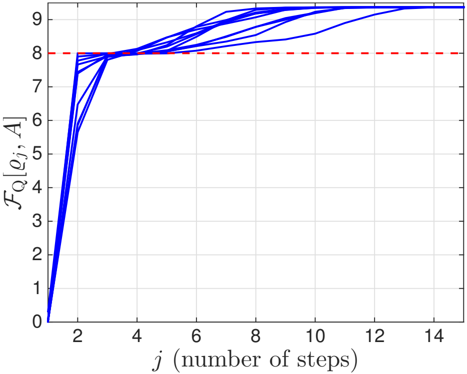
$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Convergence of the method

The precision cannot get worse with the iteration!

Convergence of the method II



Generation of the 4×4 bound entangled state.

(blue) 10 attempts. After 15 steps, the algorithm converged.

(red) Maximal quantum Fisher information for separable states.

Robustness of the states

$$\varrho(p) = (1-p)\varrho + p\varrho_{\rm noise}$$

- Robustness of entanglement: the maximal p for which $\varrho(p)$ is entangled for any separable $\varrho_{\rm noise}$.
  [Vidal and Tarrach, PRA 59, 141 (1999).]
- Robustness of metrological usefulness: the maximal p for which $\varrho(p)$ outperforms separable state for any separable $\varrho_{\rm noise}$.

Robustness of the states II

System	A	$\mathcal{F}_{\rm Q}[\varrho, A]$	$\mathcal{F}_{\rm Q}^{\rm (sep)}$	$p_{\rm white\,noise}$
four qubits	J_z	4.0088	4	0.0011
three qubits	$j_z^{(1)} + j_z^{(2)}$	2.0021	2	0.0005
2×4 (three qubits, only $1:23$ is PPT)	$j_z^{(1)} + j_z^{(2)}$	2.0033	2	0.0008

Multiqubit states

Robustness of the states III

d	$\mathcal{F}_Q[\varrho, A]$	$p_{\text{white noise}}$	$p_{\text{noise}}^{\text{LB}}$
3	8.0085	0.0006	0.0003
4	9.3726	0.0817	0.0382
5	9.3764	0.0960	0.0361
6	10.1436	0.1236	0.0560
7	10.1455	0.1377	0.0086
8	10.6667	0.1504	0.0670
9	10.6675	0.1631	0.0367
10	11.0557	0.1695	0.0747
11	11.0563	0.1807	0.0065
12	11.3616	0.1840	0.0808

- $d \times d$ systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator A is not the usual J_z.

Robustness of the states IV: 4×4 bound entangled PPT state

Let us define the following six states
$|\Psi_1\rangle = (|0,1\rangle + |2,3\rangle)/\sqrt{2}$, $|\Psi_2\rangle = (|1,0\rangle + |3,2\rangle)/\sqrt{2}$,
$|\Psi_3\rangle = (|1,1\rangle + |2,2\rangle)/\sqrt{2}$, $|\Psi_4\rangle = (|0,0\rangle + |3,3\rangle)/\sqrt{2}$,
$|\Psi_5\rangle = (1/2)(|0,3\rangle + |1,2\rangle) + |2,1\rangle/\sqrt{2}$,
$|\Psi_6\rangle = (1/2)(-|0,3\rangle + |1,2\rangle) + |3,0\rangle/\sqrt{2}$.

Our state is a mixture

$$\varrho_{4\times 4} = p\sum_{n=1}^{4} |\Psi_n\rangle\langle\Psi_n| + q\sum_{n=5}^{6} |\Psi_n\rangle\langle\Psi_n|,$$

where $q = (\sqrt{2}-1)/2$ and $p = (1-2q)/4$. We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where $H = \mathrm{diag}(1, 1, -1, -1)$.

Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with given minimal eigenvalues of the partial transpose, or for a given negativity.

[G. Vidal and R. F. Werner, PRA 65, 032314 (2002).]

Entanglement

Bipartite state	Entanglement
3×3	0.0003
4×4	0.0147
5×5	0.0239
6×6	0.0359
7×7	0.0785
UPB 3×3	0.0652
Breuer 4×4	0.1150

Convex roof of the linear entanglement entropy. The entanglement is also shown for the 3×3 state based on unextendible product bases (UPB) and for the Breuer state with a parameter $\lambda = 1/6$.

[G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015).]

Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose are useful for metrology,

Phys. Rev. Lett. 120, 020506 (2018).

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