

# Quantum states with a positive partial transpose are useful for metrology

Géza Tóth<sup>1,2,3,4</sup> and Tamás Vértesi<sup>5</sup>



<sup>1</sup>Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain

<sup>2</sup>Donostia International Physics Center (DIPC), San Sebastián, Spain

<sup>3</sup>IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

<sup>4</sup>Wigner Research Centre for Physics, Budapest, Hungary

<sup>5</sup>Institute for Nuclear Research, Hungarian Academy of Sciences, Debrecen, Hungary

Institute for Theoretical Physics,  
University of Cologne,  
12 November 2019

## 1 Motivation

- What are entangled states useful for?

## 2 Background

- Quantum Fisher information
- Recent findings on the quantum Fisher information

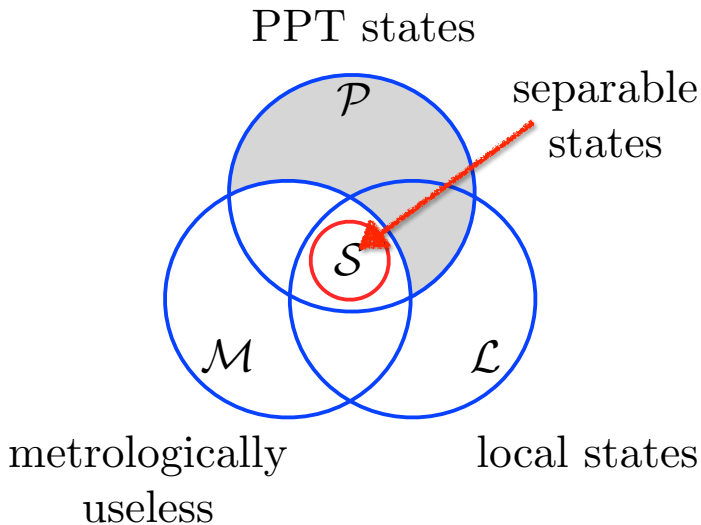
## 3 Maximizing the QFI for PPT states

- Results so far
- Our results

# What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

# What are entangled states useful for?



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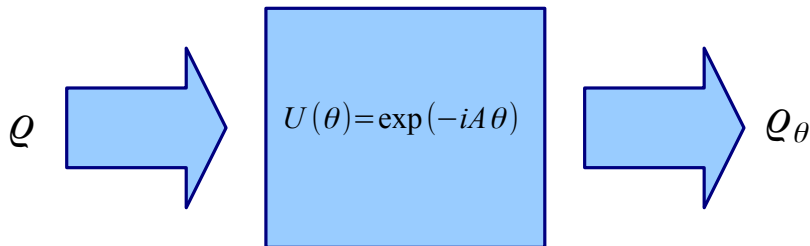
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# Quantum metrology

- Fundamental task in metrology



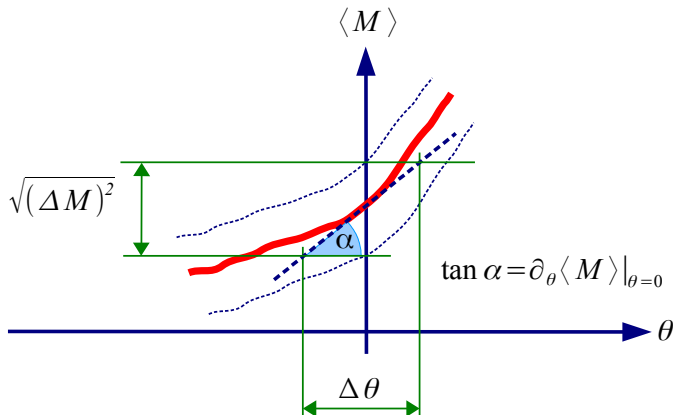
- We have to estimate  $\theta$  in the dynamics

$$U = \exp(-iA\theta).$$

# Precision of parameter estimation

- Measure an operator  $M$  to get the estimate  $\theta$ . The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



# The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, \mathbf{A}]}, \quad (\Delta\theta)^{-2} \leq F_Q[\varrho, \mathbf{A}].$$

where  $F_Q[\varrho, \mathbf{A}]$  is the **quantum Fisher information**.

- The quantum Fisher information is

$$F_Q[\varrho, \mathbf{A}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | \mathbf{A} | l \rangle|^2,$$

where  $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ .

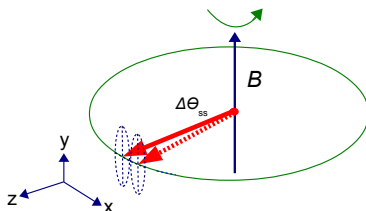
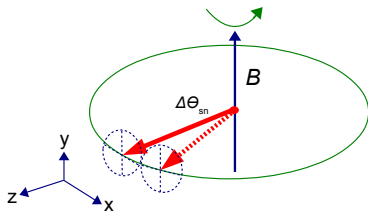


# Special case $A = J_l$

- The operator  $A$  is defined as

$$A = J_l = \sum_{n=1}^N j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer



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# Properties of the Fisher information

Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance,  
$$F[|\psi\rangle\langle\psi|, A] = 4(\Delta A)^2_\psi.$$
- For mixed states, it is convex.

# The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N, \quad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most  $k$ -particle entanglement ( $k$  is divisor of  $N$ )

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

# Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\rho_k, \Psi_k} \sum_k \rho_k (\Delta A)^2_k,$$

where

$$\varrho = \sum_k \rho_k |\Psi_k\rangle \langle \Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);  
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Extended convexity for non-unitary dynamics.

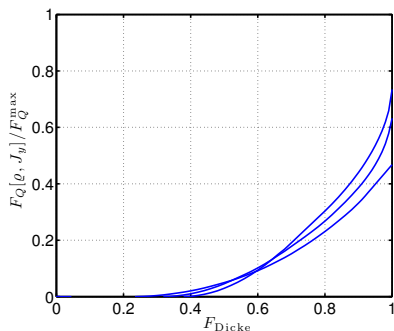
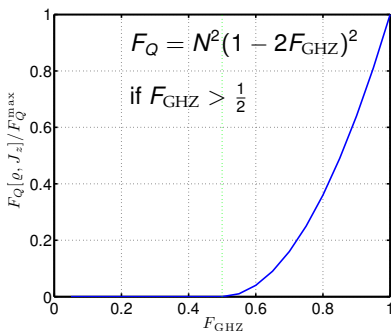
[S. Alipour, A. T. Rezakhani, Phys. Rev. A 91, 042104 (2015).]

- Convex roof over purifications.

[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

# Witnessing the quantum Fisher information based on few measurements

- Let us bound the quantum Fisher information based on some measurements.



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for  $N = 4, 6, 12$ .

# Continuity of QFI and QFI for symmetric states

- Arbitrarily small entanglement can be used to get close to Heisenberg scaling.
- The difference between the QFI of two states can be bounded by the distance of the two states.
- Bound on the QFI with the geometric measure of entanglement.

[R. Augusiak, J. Kołodyński, A. Streltsov, M. N. Bera, A. Acín, M. Lewenstein, PRA 2016]

- Continuity in the non-unitary case:

[A. T. Rezakhani, S. Alipour, M. Hassani, PRA 2019]

- Random pure states of distinguishable particles typically do not lead to super-classical scaling of precision.
- Random states from the symmetric subspace typically achieve the optimal Heisenberg scaling.

[M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acín, M. Lewenstein, PRX 2016]

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# Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.  
[ P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012). ]
- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.  
[ Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015). ]

on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

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# Our results

We look for bipartite PPT entangled states and multipartite states that are PPT with respect to all partitions.

# Maximizing the QFI for PPT states: brute force

- Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where  $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ .

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.
- Not guaranteed to find the global maximum.
- Note: Finding the *minimum* is possible!

# Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2} = \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2} \quad (\text{dynamics is } U = e^{-iA\theta}).$$

- The bound is sharp

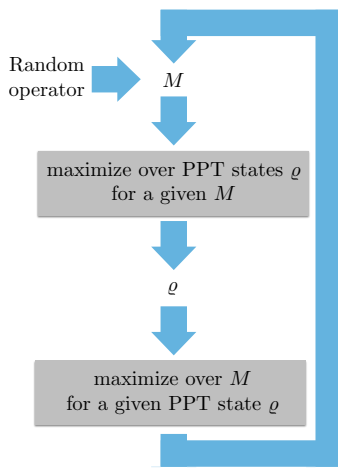
$$F_Q[\varrho, A] = \max_M \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2}.$$

[ M. G. Paris, *Int. J. Quantum Inform.* 2009. Used, e.g., in F. Fröwis, R. Schmied, and N. Gisin, 2015; I. Appelaniz *et al.*, *NJP* 2015.]

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2}.$$

# Saw-saw algorithm for maximizing the precision



Similar iterative approach was used for maximizing over  $\varrho$  for noisy states: [ [Macieszczak, arXiv:1312.1356v1](#);

[Macieszczak, Fraas, Demkowicz-Dobrzanski, NJP 2014. \]](#)

# Maximize over PPT states for a given $M$

Best precision for PPT states for a given operator  $M$  can be obtained by a semidefinite program.

*Proof.*—Let us define first

$$\begin{aligned} f_M(X, Y) = \min_{\rho} \quad & \text{Tr}(M^2 \rho), \\ \text{s.t.} \quad & \rho \geq 0, \rho^{\text{Tk}} \geq 0 \text{ for all } k, \text{Tr}(\rho) = 1, \\ & \langle i[M, A] \rangle = X \text{ and } \langle M \rangle = Y. \end{aligned}$$

The best precision for a given  $M$  and for PPT states is

$$(\Delta\theta)^2 = \min_{X, Y} \frac{f_M(X, Y) - Y^2}{X^2}.$$

The state giving the best precision is  $\rho_{\text{PPTopt}}$ .

## Maximize over $M$ for a given PPT state

For a state  $\varrho$ , the best precision is obtained with the operator given by the symmetric logarithmic derivative

$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle \langle l| \langle k|A|l\rangle,$$

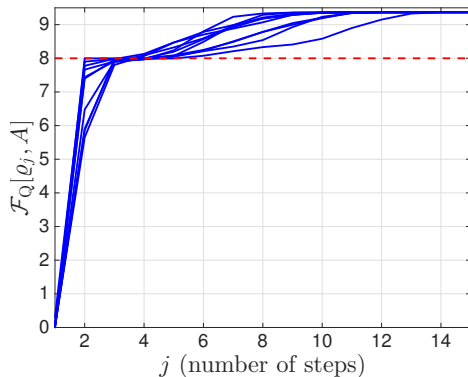
where  $\varrho = \sum_k \lambda_k |k\rangle \langle k|$ .



# Convergence of the method

The precision cannot get worse with the iteration!

# Convergence of the method II



Generation of the  $4 \times 4$  bound entangled state.

- (blue) 10 attempts. After 15 steps, the algorithm converged.
- (red) Maximal quantum Fisher information for separable states.

# Robustness of the states

$$\varrho(p) = (1 - p)\varrho + p\varrho_{\text{noise}}$$

- Robustness of entanglement: the maximal  $p$  for which  $\varrho(p)$  is entangled for any separable  $\varrho_{\text{noise}}$ .  
[ Vidal and Tarrach, PRA 59, 141 (1999). ]
- **Robustness of metrological usefulness**: the maximal  $p$  for which  $\varrho(p)$  outperforms separable state for any separable  $\varrho_{\text{noise}}$ .

# Robustness of the states II

System	$A$	$\mathcal{F}_Q[\varrho, A]$	$\mathcal{F}_Q^{(\text{sep})}$	$\rho_{\text{white noise}}$
four qubits	$J_z$	4.0088	4	0.0011
three qubits	$j_z^{(1)} + j_z^{(2)}$	2.0021	2	0.0005
$2 \times 4$ (three qubits, only 1 : 23 is PPT)	$j_z^{(1)} + j_z^{(2)}$	2.0033	2	0.0008

Multiqubit states

## Robustness of the states III

$d$	$\mathcal{F}_Q[\varrho, A]$	$\rho_{\text{white noise}}$	$\rho_{\text{noise}}^{\text{LB}}$
3	8.0085	0.0006	0.0003
4	9.3726	0.0817	0.0382
5	9.3764	0.0960	0.0361
6	10.1436	0.1236	0.0560
7	10.1455	0.1377	0.0086
8	10.6667	0.1504	0.0670
9	10.6675	0.1631	0.0367
10	11.0557	0.1695	0.0747
11	11.0563	0.1807	0.0065
12	11.3616	0.1840	0.0808

- $d \times d$  systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator  $A$  is not the usual  $J_z$ .

## Robustness of the states IV: $4 \times 4$ bound entangled PPT state

Let us define the following six states

$$|\Psi_1\rangle = (|0, 1\rangle + |2, 3\rangle)/\sqrt{2}, \quad |\Psi_2\rangle = (|1, 0\rangle + |3, 2\rangle)/\sqrt{2},$$

$$|\Psi_3\rangle = (|1, 1\rangle + |2, 2\rangle)/\sqrt{2}, \quad |\Psi_4\rangle = (|0, 0\rangle - |3, 3\rangle)/\sqrt{2},$$

$$|\Psi_5\rangle = (1/2)(|0, 3\rangle + |1, 2\rangle) + |2, 1\rangle/\sqrt{2},$$

$$|\Psi_6\rangle = (1/2)(-|0, 3\rangle + |1, 2\rangle) + |3, 0\rangle/\sqrt{2}.$$

Our state is a mixture

$$\rho_{4 \times 4} = p \sum_{n=1}^4 |\Psi_n\rangle\langle\Psi_n| + q \sum_{n=5}^6 |\Psi_n\rangle\langle\Psi_n|,$$

where  $q = (\sqrt{2} - 1)/2$  and  $p = (1 - 2q)/4$ . We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where  $H = \text{diag}(1, 1, -1, -1)$ .

# Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with given minimal eigenvalues of the partial transpose, or for a given negativity.

[ G. Vidal and R. F. Werner, PRA 65, 032314 (2002). ]

# Entanglement

Bipartite state	Entanglement
$3 \times 3$	0.0003
$4 \times 4$	0.0147
$5 \times 5$	0.0239
$6 \times 6$	0.0359
$7 \times 7$	0.0785
UPB $3 \times 3$	0.0652
Breuer $4 \times 4$	0.1150

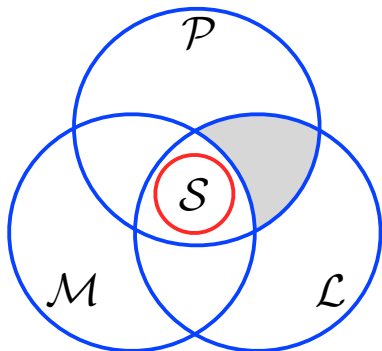
Convex roof of the linear entanglement entropy. The entanglement is also shown for the  $3 \times 3$  state based on unextendible product bases (UPB) and for the Breuer state with a parameter  $\lambda = 1/6$ .

[ G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015). ]



# Metrologically useful quantum states with LHV models (PPT)

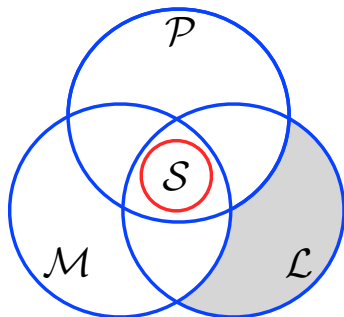
Consider the  $2 \times 4$  state listed before. Possible to construct numerically a LHV model for the state.



[ F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, PRL 2016;  
D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, PRL 2016. ]

# Metrologically useful quantum states with LHV models (non-PPT)

- Two-qubit Werner state  $p|\Psi^-\rangle\langle\Psi^-| + (1-p)\mathbb{1}/4$ , with  $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ .
- Better for metrology than separable states ( $\mathcal{F}_Q > 2$ ) for  $p > 1 - 0.3596 = 0.6404$ .
- They do not violate a Bell inequality for  $p < 0.6829$ .



# Cluster states

Cluster states: resource in measurement-based quantum computing

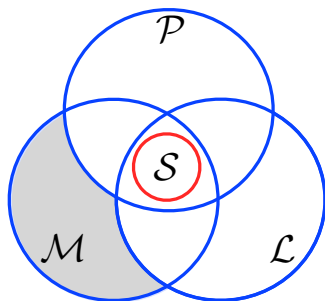
[ R. Raussendorf and H. J. Briegel, PRL 2001. ]

- Fully entangled pure states.
- Violate a Bell inequality

[ V. Scarani, A. Acín, E. Schenck, M. Aspelmeyer, PRA 2005; O. Gühne, GT, P. Hyllus, H. J. Briegel, PRL 2005; GT, O. Gühne, and H. J. Briegel, PRA 2006. ]

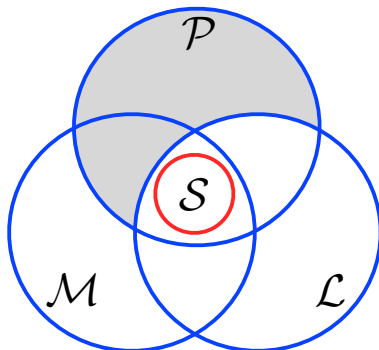
- Metrologically not useful

[ P. Hyllus, O. Gühne, and A. Smerzi, PRA 2010. ]



# Non-local PPT states

Counterexample for the Peres conjecture



[ T. Vértesi and N. Brunner, Nature Communications 2015. ]

# Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose  
are useful for metrology,

[Phys. Rev. Lett. 120, 020506 \(2018\).](#)

THANK YOU FOR YOUR ATTENTION!

