How long does it take to obtain a physical density matrix?

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Murmansk
Tromsø
Kiruna
White Sea
Arkhangel
Norwegian Sea
Reykjavík
ICELAND
Arctic Circle
NORWAY
Luleå
Oulu
Umeå
FINLAND
Lake Onega
Trondheim
SWEDEN
Gulf of Bothnia
Tampere
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
Helsinki
Turku
Gävle
St. Petersburg
SHETLAND ISLANDS
Bergen
Oslo
ALAND ISLANDS
Gulf of Finland
Tallinn
ESTONIA
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ORKNEY ISLANDS
Stockholm
HEBRIDES
Stavanger
Skagerrak
Kattegat
Göteborg
Gotland
LATVIA
Riga
Aberdeen
North Atlantic Ocean
North Sea
Glasgow
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Sarajevo
SERBIA AND MONTENEGRO
BULGARIA
Toulouse
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SAN MARINO
Sofia
Andorra la Vella
Marseille
Florence
Ligurian Sea
Porto
ANDORRA
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ITALY
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ALB.
Thessaloniki
TURK
Lisbon
Tagus
Balearic Sea
Naples
SPAIN
Valencia
Sardinia
Tyrrhenian Sea
GREECE
Aegean Sea
İzmir
BALEARIC ISLANDS
Athens
Sevilla
Cagliari
Ionian Sea
Mediterranean Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Rhodes
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
300 Kilometers
300 Miles
Boundary representation not necessarily authoritative

Outline

1. **Motivation**
   - Why quantum tomography is important?
2. **Quantum experiments with multi-qubit systems**
   - Physical systems
   - Local measurements
3. **Full quantum state tomography**
   - Basic ideas and scaling
   - Experiments
4. **Obtaining a physical density matrix**
   - The usual fitting procedure to obtain a physical matrix
   - Our insight: the key are the eigenvalues
   - Our proposal: Hypothesis testing

Why tomography is important?

- Many experiments aiming to create many-body entangled states.
- Quantum state tomography can be used to check how well the state has been prepared.
- However, there are some long standing problems:
    - the number of measurements scales exponentially with the number of qubits,
    - the “raw” density matrices obtained from state tomography is not physical, etc.

Outline

Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions
  T. Monz, P. Schindler, J.T. Barreiro, M. Chwalla, D. Nigg, W.A. Coish, M. Harlander, W. Haensel, M. Hennrich, R. Blatt, Phys. Rev. Lett. 106, 130506 (2011).

- 10 qubits with photons
  W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, J.-W. Pan, Nature Physics, 6, 331 (2010).

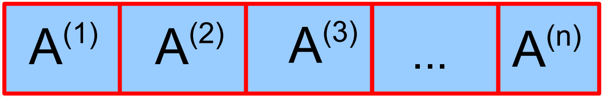
Outline

Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.



- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

Outline

Full quantum state tomography

- The density matrix can be reconstructed from 3^n measurement settings.

Example

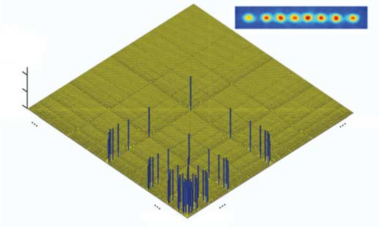
For $n = 4$, the measurements are

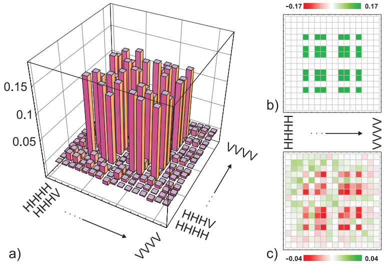
1.	X	X	X	X
2.	X	X	X	Y
3.	X	X	X	Z
		...		
3^4.	Z	Z	Z	Z

- Note again that the number of measurements scales exponentially in n.

Outline

Experiments with ions and photons





- 8 ions: H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).
- 4 photons: N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. 98, 063604 (2007).
- 6 photons: C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Phys. Rev. Lett. 113, 040503 (2014).

Outline

Obtain a density matrix

- The density matrix can be decomposed into correlations as

$$\varrho = \frac{1}{2^n} \sum_{\boldsymbol{\mu}} T_{\boldsymbol{\mu}} \sigma_{\boldsymbol{\mu}},$$

  where

$$\sigma_{\boldsymbol{\mu}} = \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \cdots \otimes \sigma_{\mu_n},$$

  $\mu_i \in \{0, 1, 2, 3\}$, and σ_0 denotes the identity matrix.

- The correlation matrix is defined as

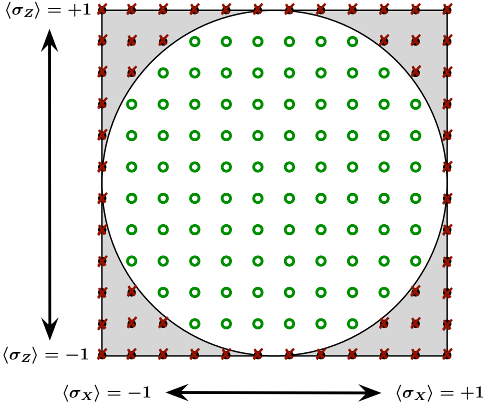
$$T_{\boldsymbol{\mu}} = \langle \sigma_{\boldsymbol{\mu}} \rangle.$$

Obtain a density matrix II

- How can we obtain the estimate $\tilde{\varrho}$? We just measure $T_{\boldsymbol{\mu}} = \langle \sigma_{\boldsymbol{\mu}} \rangle$. (linear inversion)
- Problem: we have finite number of measurements.
- Hence, we cannot get $\langle \sigma_{\boldsymbol{\mu}} \rangle$ exactly and our state will not be physical.
- The consequence is not only a small error, but that $\varrho \geq 0$ is not fulfilled.

Obtain a density matrix III

- 1 qubit, 11 measurements.



Obtain a density matrix IV

The negative eigenvalues are

- due to finite statistics,
- are not due to some experimental error.

Why negative eigenvalues are a problem?

We cannot calculate

- fidelities with a mixed state,
- entropies,
- purity,
- entanglement, etc.

We can still calculate

- the fidelity with a pure state (=expectation value of a projector).

Fitting

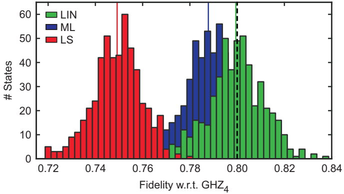
- Method to get rid of the negative eigenvalues of ϱ.
- Find the physical density matrix in a best agreement with the experimental data.
- Main methods: an elegant theory using maximum likelihood, least squares.

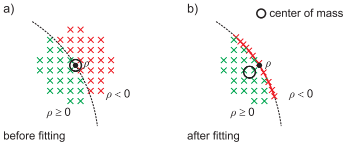
[Z. Hradil, Phys. Rev. A 55, R1561 (1997); D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A 64, 052312 (2001); R. Blume- Kohout, New J. Phys. 12, 043034 (2010).]

Problems with fitting

- Bias: ML is unbiased for infinite statistics. We have finite data.

- Fidelity changes, detection of fake entanglement



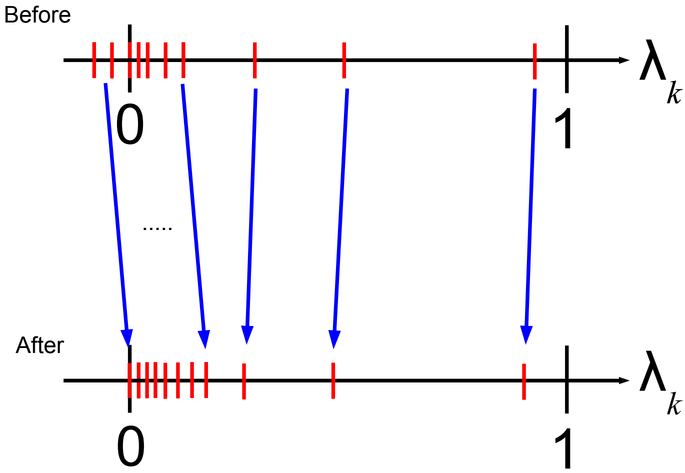


- Unfortunately, the fidelity with respect to the pure state we wanted to prepare also changes.

[Schwemmer et al., PRL 114, 080403 (2015).]

Problems with fitting II

Another artifact: About half of the eigenvalues become zero.



Small eigenvalues increase   Large eigenvalues decrease

Heuristic explanation of the problem

Fitting is

- Like frequency analysis.
- Large frequencies do not matter if we have finite data.

  (Harald Weinfurter)

- Like the police is looking for someone, and the search warrant says that the person is 3 meter tall.
- Distance to non-physical states can have various definitions.

  (GT)

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   - Our proposal: Hypothesis testing

Let us analyze the problem

- Completely mixed state

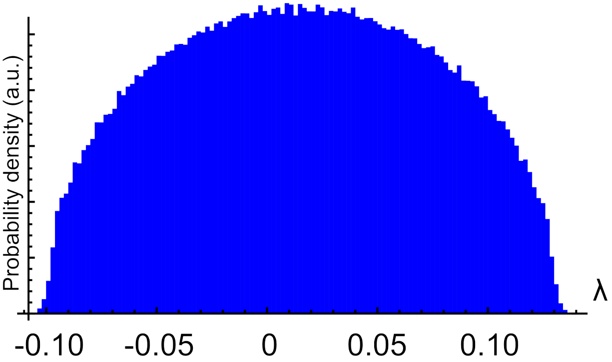
$$\varrho_{\mathrm{wn}} = \frac{1}{2^n}\sigma_{0,0,\ldots,0} = \frac{1}{2^n}\mathbb{1}$$

with 2^n degenerate eigenvalues $\lambda_i = 1/2^n$.

- We use overcomplete tomography, which is based on measuring the Pauli correlations.

Distribution of eigenvalues

- Consider $n = 6$ qubit maximally mixed state
- Simulate $N = 100$ measurements per setting
- Estimate density matrix
- Repeat 10000 times
- Histogram of eigenvalues



Distribution of eigenvalues

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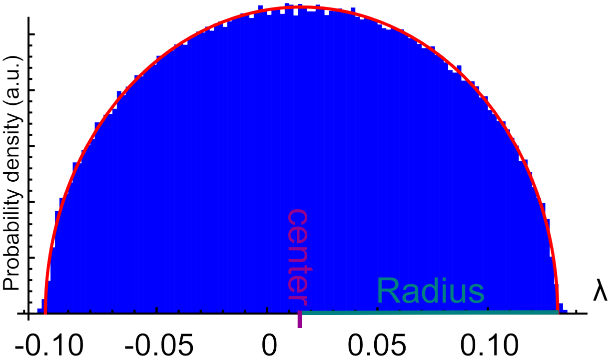


Distribution of eigenvalues II

The Wigner semicircle appears also in random matrix for a special type of matrices

- all elements have the same Gaussian distribution,
- they are not correlated with each other.

Now the elements of the density matrix obtained from tomography

- have different variances,
- they are even correlated with each other.

Highly non-trivial result: We prove that we obtain the Wigner semicircle distribution for the usual, overcomplete tomography.

Derivation (slide from Lukas Knips)

- Concept: compare moments of eigenvalue distribution to moments of ideal semicircle function

- Define semicircle distribution

$$f_{c,R}(x) = \underbrace{\frac{2}{\pi R^2}}_{Z} \sqrt{(x-c)^2 - R^2}$$

with (even) moments

$$m_2^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^2 x = \left(\frac{R}{2}\right)^2,$$
$$m_4^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^4 x = 2\left(\frac{R}{2}\right)^4,$$
$$m_6^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^6 x = 5\left(\frac{R}{2}\right)^6,$$
$$m_8^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^8 x = 14\left(\frac{R}{2}\right)^8.$$

Using the *Catalan* numbers

$$\mathcal{C}_{j+1} = \mathcal{C}_j \frac{2(2j+1)}{j+2}$$

we obtain

$$m_{2k}^{\mathrm{sc}} = \int_{-\infty}^{\infty} f_{0,R}(x)\, x^{2k} x = \mathcal{C}_k \left(\frac{R}{2}\right)^{2k}$$

- Odd (centralized) moments vanish

- Goal: reproduce Catalan numbers in distribution of eigenvalues

- Calculate all moments of eigenvalue distribution:

$$\begin{aligned} m_k^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n} \mathbb{E}\left[\lambda_i^k\right] \\ &= \frac{1}{2^n}\mathbb{E}\left[\sum_{i=1}^{2^n} \lambda_i^k\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\operatorname{Tr}\left(D^k\right)\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\operatorname{Tr}\left(\left(U^\dagger \varrho U\right)^k\right)\right] \\ &= \mathbb{E}\left[\frac{1}{2^n}\operatorname{Tr}\left(\varrho^k\right)\right] \end{aligned}$$

- Second moment of (centered) distribution:

$$\begin{aligned} m_2^{\mathrm{ev}} &= \frac{1}{2^{3n}}\sum_{\vec{\mu},\vec{\nu}} \mathbb{E}\left[T_{\vec{\mu}} T_{\vec{\nu}}\right] 2^n \delta_{\vec{\mu},\vec{\nu}} \\ &= \frac{2^n}{2^{3n}} \sum_{\vec{\mu}} \mathbb{E}\left[T_{\vec{\mu}}^2\right] \end{aligned}$$

overcomplete Pauli scheme:

$$\begin{aligned} m_2^{\mathrm{ev}} &= \frac{1}{4^n N}\sum_{j=0}^{n-1}\binom{n}{j}\frac{3^{n-j}}{3^j} \\ &= \frac{10^n - 1}{12^n}\frac{1}{N}. \end{aligned}$$

with n qubits, N events per basis element.

- Comparision of m_2^{sc}, m_2^{ev} yields:

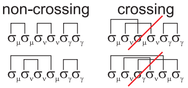
$$R = 2\sqrt{\frac{10^n-1}{12^n}}\frac{1}{\sqrt{N}}$$

- Fourth moment:

$$\begin{aligned} m_4^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^4\right] \\ &= \frac{1}{2^{5n}}\sum_{\vec{\mu},\vec{\nu},\vec{\gamma},\vec{\lambda}} \mathbb{E}\left[T_{\vec{\mu}}T_{\vec{\nu}}T_{\vec{\gamma}}T_{\vec{\lambda}}\right] \\ &\qquad \cdot \operatorname{Tr}\left(\sigma_{\vec{\mu}}\sigma_{\vec{\nu}}\sigma_{\vec{\gamma}}\sigma_{\vec{\lambda}}\right) \\ &= \frac{1}{2^{5n}}\frac{1}{2!}\sum_{\vec{\mu}}\sum_{\vec{\nu}:\{\vec{\nu}\neq\vec{\mu}\}} \mathbb{E}\left[T_{\vec{\mu}}^2 T_{\vec{\nu}}^2\right] \\ &\qquad \cdot \operatorname{Tr}\left(\sum_{i=1}^{6}\mathcal{P}_i\left(\sigma_{\vec{\mu}}\sigma_{\vec{\mu}}\sigma_{\vec{\nu}}\sigma_{\vec{\nu}}\right)\right) \end{aligned}$$

- Sixth moment:

$$\begin{aligned} m_6^{\mathrm{ev}} &= \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^6\right] \\ &\approx \frac{1}{2^{7n}}\frac{1}{3!}\sum_{\vec{\mu}}\sum_{\vec{\nu}:\{\vec{\nu}\neq\vec{\mu}\}}\sum_{\vec{\gamma}:\{\vec{\gamma}\neq\vec{\nu},\vec{\gamma}\neq\vec{\mu}\}} \\ &\qquad \mathbb{E}\left[T_{\vec{\nu}}^2 T_{\vec{\mu}}^2 T_{\vec{\gamma}}^2\right] \cdot \\ &\qquad \operatorname{Tr}\left(\sum_{i=1}^{90}\mathcal{P}_i\left(\sigma_{\vec{\mu}}\sigma_{\vec{\mu}}\sigma_{\vec{\nu}}\sigma_{\vec{\nu}}\sigma_{\vec{\gamma}}\sigma_{\vec{\gamma}}\right)\right) \end{aligned}$$

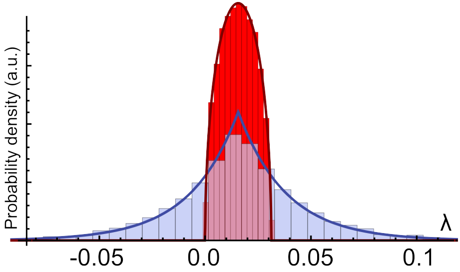


- Only non-crossing partitions (amount given by *Catalan* numbers) contribute:

$$m_k^{\mathrm{ev}} = \frac{1}{2^n}\sum_{i=1}^{2^n}\mathbb{E}\left[\lambda_i^{2k}\right] = \frac{\mathcal{C}_k}{N^k} \qquad \square$$

Other type of tomography

- Not all tomographies lead to a Wigner semicircle



[F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A 64, 052312 (2001).]

How long do we have to measure to get a physical state?

- Pure state mixed with white noise

$$\varrho_q = q|\psi\rangle\langle\psi| + (1-q)\varrho_{\rm cm}.$$

- The center of the semicircle is shifted to

$$c_q = \frac{1-q}{2^n - r}.$$

- The radius of the semicircle is

$$R = 2\sqrt{\frac{10^n - 1}{12^n}}\frac{1}{\sqrt{N}} \approx 2\left(\frac{5}{6}\right)^{\frac{n}{2}}\frac{1}{\sqrt{N}}.$$

- Physical ϱ if

$$R \le c_q \Rightarrow N \ge N_0 = 4\left(\frac{5}{6}\right)^n\left(\frac{2^n - 1}{1-q}\right)^2.$$

How long do we have to measure to get a physical state? II

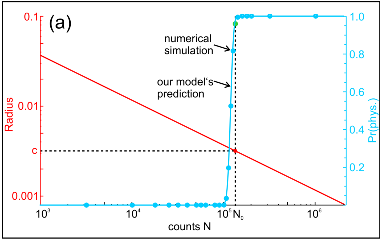
- The minimum number of measurements needed is

$$N_0 = 4\left(\frac{5}{6}\right)^n \left(\frac{2^n - 1}{1 - q}\right)^2,$$

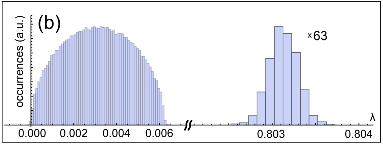
where n is the number of qubits.

How long do we have to measure to get a physical state? III

- Six-qubit GHZ state mixed with $q = 0.2$ white noise, radius and probability of physical matrix



- The eigenvalues of the noisy state at $N = N_0$ measurements



Outline

Hypothesis testing

- Assume
$$\varrho = p\varrho_r + (1-p)\frac{\mathbb{1}}{2^n},$$
where ϱ_r is a pure of a low rank state.

- Colored noise? We cannot tell, since the Wigner semicircle will cover all structure.

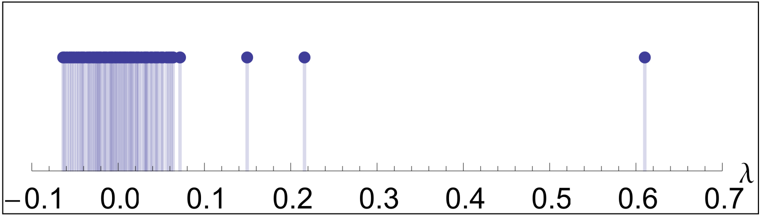
- Hypothesis about how many eigenvalues are “real”.

Hypothesis testing

- We prepare experimentally a six-qubit Dicke state

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + ... + |111000\rangle).$$

- Quantum state tomography with around 230 events per setting.
- Hypothesis: 3 eigenvalues + noise. Is this correct?



Hypothesis testing II

- We prepare experimentally a six-qubit Dicke state

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + ... + |111000\rangle).$$

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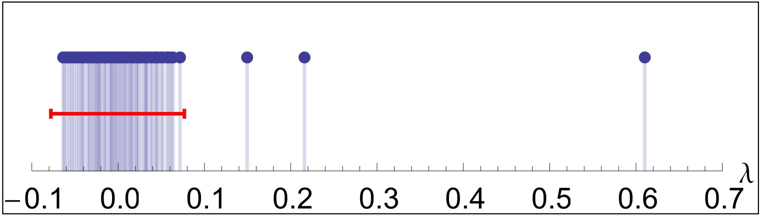

Hypothesis testing III

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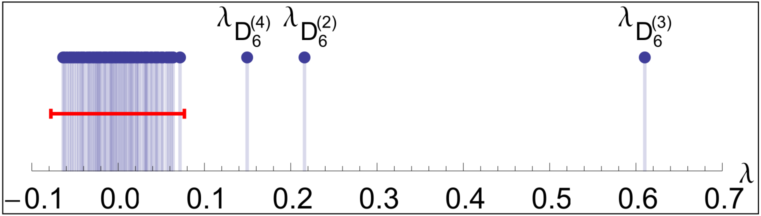
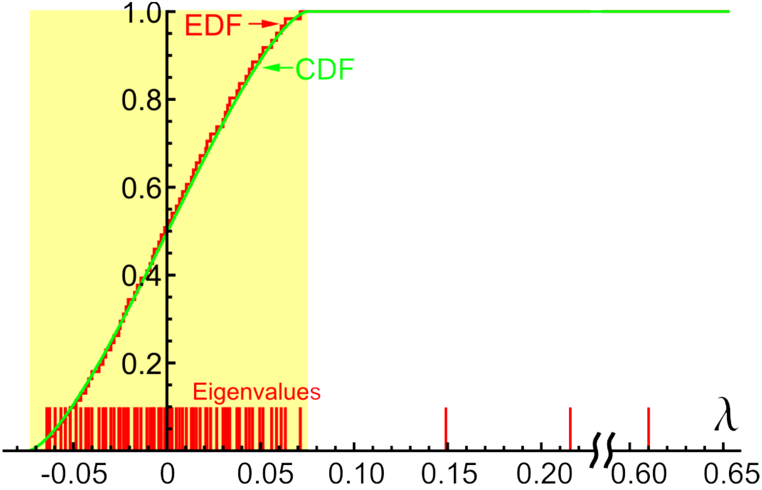


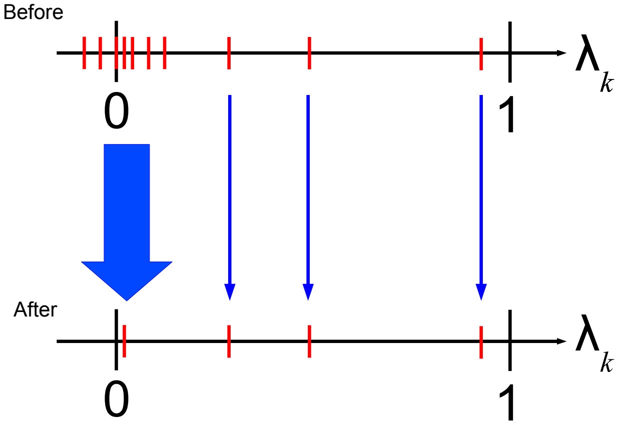
Is the hypothesis correct?

- Empirical distribution function (EDF) vs. Cumulative distribution function (CDF) of the Wigner semicircle



Our method in a single figure

Large, useful eigenvalues are not affected!

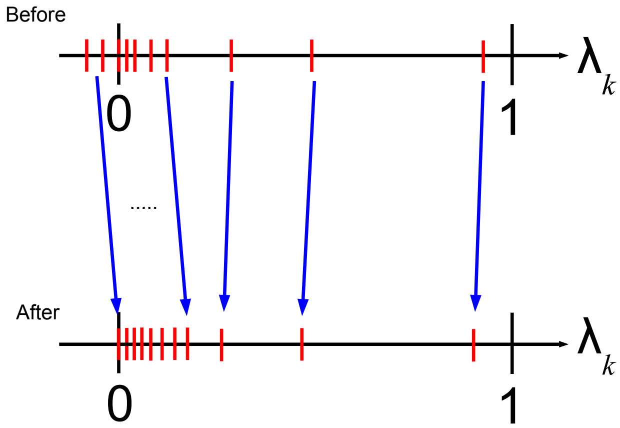


Small eigenvalues are replaced by their average

Large eigenvalues do not change

Just to compare: old method

Large, useful eigenvalues are affected!



Small eigenvalues increase

Large eigenvalues decrease

Summary

- We discussed the distribution of the eigenvalues of density matrices obtained from tomography.
- We proposed a simple solution for a long standing problem, namely, getting rid of the negative eigenvalues.
- I thank Lukas Knips for most of the figures for this talk.

See:
L. Knips, C. Schwemmer, N. Klein, J. Reuter, G. Tóth, and H. Weinfurter,

How long does it take to obtain a physical density matrix?, arxiv:1512.06866.

THANK YOU FOR YOUR ATTENTION!