# Quantum metrology from a quantum information science perspective


Géza Tóth[1,2,3] and Iagoba Apellaniz[1,4]

[1] Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
[2] Donostia International Physics Center (DIPC), San Sebastián, Spain
[3] IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[4] Wigner Research Centre for Physics, Budapest, Hungary
[4] Mondragon University, Mondragon, Spain


Quantum Science Seminar, Budapest
25 February 2021

# Outline

1. **Motivation**
   - Why is quantum metrology interesting?
2. **Simple examples of quantum metrology**
   - Classical case: Clock arm
   - Quantum case: Single spin-1/2 particle
   - Magnetometry with the fully polarized state
   - Magnetometry with the spin-squeezed state
   - Metrology with the GHZ state
   - Interferometry with squeezed photonic states
3. **Entanglement theory**
   - Multipartite entanglement
   - The spin-squeezing criterion
4. **Quantum metrology using the quantum Fisher information**
   - Quantum Fisher information
   - Quantum Fisher information in linear interferometers
   - Noise and imperfections

# Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum systems, i.e., in cold gases, trapped cold ions or photons.

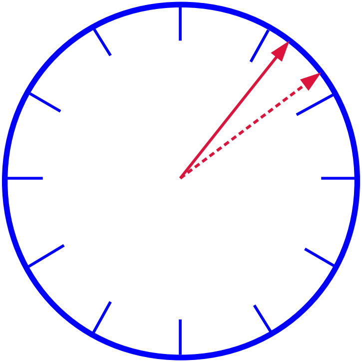
- Can such quantum systems outperform classical systems in something useful, i.e., metrology?

# Outline

1. Motivation
   - Why is quantum metrology interesting?
2. Simple examples of quantum metrology
   - Classical case: Clock arm
   - Quantum case: Single spin-1/2 particle
   - Magnetometry with the fully polarized state
   - Magnetometry with the spin-squeezed state
   - Metrology with the GHZ state
   - Interferometry with squeezed photonic states
3. Entanglement theory
   - Multipartite entanglement
   - The spin-squeezing criterion
4. Quantum metrology using the quantum Fisher information
   - Quantum Fisher information
   - Quantum Fisher information in linear interferometers
   - Noise and imperfections

# Classical case: Estimating the angle of a clock arm

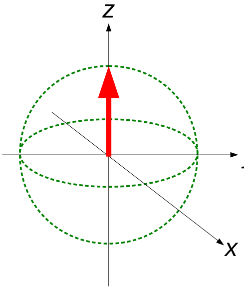
- Arbitrary precision ("in principle").



# Outline

# Quantum case: A single spin-$1/2$ particle

- Spin-1/2 particle polarized in the *z* direction.



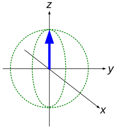

- We measure the spin components.

$$j_x \begin{array}{l} \nearrow +1/2, \;\; 50\% \\ \searrow -1/2, \;\; 50\% \end{array} \qquad j_y \begin{array}{l} \nearrow +1/2, \;\; 50\% \\ \searrow -1/2, \;\; 50\% \end{array} \qquad j_z \rightarrow +1/2$$

# Quantum case: A single spin-$1/2$ particle II

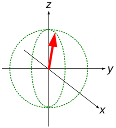
- We cannot measure the three spin coordinates exactly $j_x, j_y, j_z$.
- In quantum physics, we can get only discrete outcomes in measurement. In this case, $+1/2$ and $-1/2$.
- A single spin-1/2 particle is not a good clock arm.

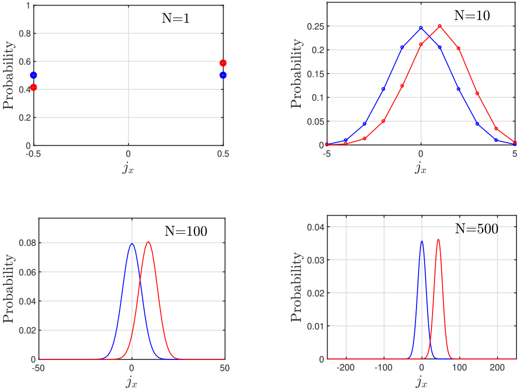
# Several spin-1/2 particles




$10^{o}$ rotation around $y$

$\Rightarrow$



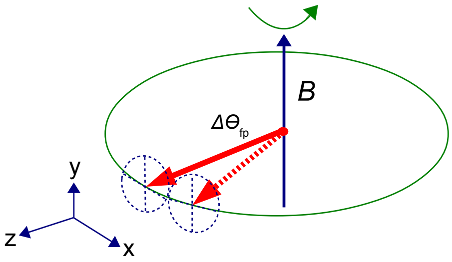



# Outline

# Magnetometry with the fully polarized state

- $N$ spin-$1/2$ particles, all fully polarized in the $z$ direction.

- Magtetic field $B$ points to the $y$ direction.




- Note the uncertainty ellipses. $\Delta\theta_{\mathrm{fp}}$ is the minimal angle difference we can measure.

# Magnetometry with the fully polarized state II

- Collective angular momentum components

$$J_l := \sum_{n=1}^{N} j_l^{(n)}$$

for $l = x, y, z$, where $j_l^{(n)}$ are single particle operators.

- Dynamics

$$|\Psi\rangle = U_\theta |\Psi_0\rangle, \qquad U_\theta = e^{-iJ_y\theta},$$

where $\hbar = 1$.

- Rotation around the $y$-axis.

# Magnetometry with the fully polarized state III

- In order to see the full picture, we need to consider $\nu$ measurements of $M$.

- We have to look for the average of the measured values

$$\overline{\mu}_\nu = \frac{1}{\nu}\sum_{n=1}^{\nu}\mu_k.$$

- Let us consider the $P(\mu|\theta)$ probability distribution. With that

$$\overline{\mu} = \int P(\mu|\theta)\mu d\mu, \quad (\Delta\mu)^2 = \int P(\mu|\theta)(\mu - \overline{\mu})^2 d\mu.$$

[ L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, "Quantum metrology with nonclassical states of atomic ensembles," Rev. Mod. Phys. 90, 035005 (2018). ]

# Magnetometry with the fully polarized state IV

- The estimate is the $\theta$ for which $\overline{\mu} = \overline{\mu}_\nu$.

- The achievable precision for the estimator is

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta\mu)^2}{|\partial_\theta \overline{\mu}|^2},$$

  which can be reached if $P(\mu|\theta)$ is Gaussian and $d\Delta\mu/d\theta \ll d\overline{\mu}/d\theta$.

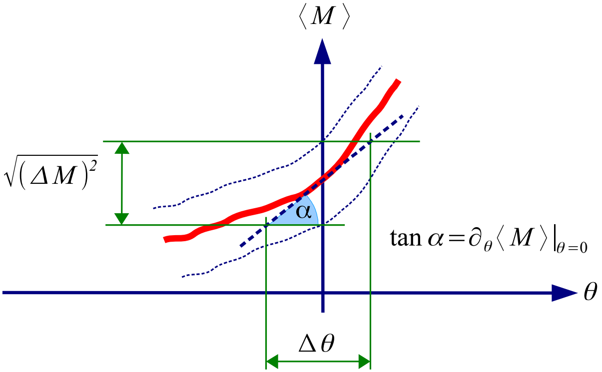
- With quantum mechanical quantites

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta M)^2}{|\partial_\theta \langle M\rangle|^2} \equiv \frac{1}{\nu}(\Delta\theta)^2_M.$$

# Magnetometry with the fully polarized state V

- Interpretation of the error propagation formula:

$$(\Delta\theta)^2_M = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



# Magnetometry with the fully polarized state VI

- We consider the fully polarized states of $N$ spin-1/2 particles

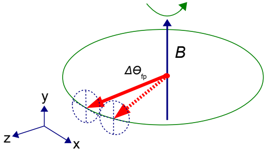
$$|+\tfrac{1}{2}\rangle^{\otimes N}.$$

- For this state,

$$\langle J_z\rangle = N/2, \quad \langle J_x\rangle = 0, \quad (\Delta J_x)^2 = N/4.$$

- We measure the operator

$$M = J_x.$$




- Expectation value and variance

$$\begin{aligned}
\langle M\rangle(\theta) &= \langle J_z\rangle\sin(\theta) + \langle J_x\rangle\cos(\theta), \\
(\Delta M)^2(\theta) &= (\Delta J_x)^2\cos^2(\theta) + (\Delta J_z)^2\sin^2(\theta) \\
&+ \left(\tfrac{1}{2}\langle J_xJ_z + J_zJ_x\rangle - \langle J_x\rangle\langle J_z\rangle\right)\sin(2\theta).
\end{aligned}$$

# Magnetometry with the fully polarized state VII

- It is not like a classical clock arm, for $\theta = 0$ we have a nonzero uncertainty

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta M)^2}{|\partial_\theta \langle M\rangle|^2} = \frac{1}{\nu}\frac{(\Delta J_x)^2}{\langle J_z\rangle^2} = \frac{1}{\nu}\frac{1}{N}.$$

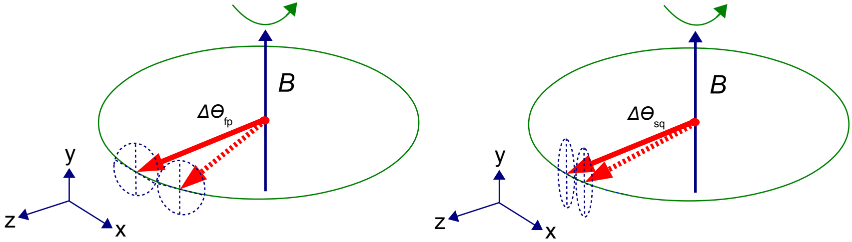
- In some cold gas experiment, we can have $10^3 - 10^{12}$ particles.

- Later we will see that with a separable quantum state we cannot have a better precision.

# Outline

# Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing




fully polarized state (fp)

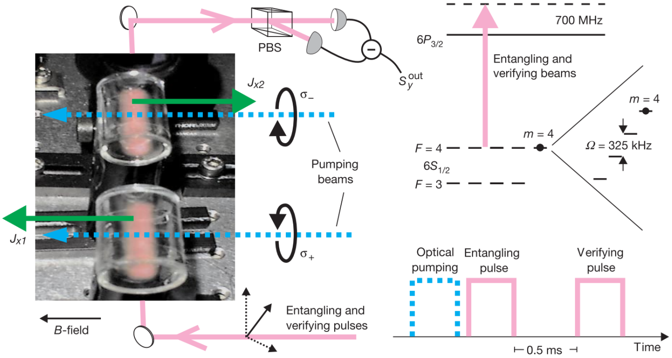
spin-squeezed state (sq)

$\Delta\theta_{\mathrm{fp}}$ and $\Delta\theta_{\mathrm{sq}}$ are the minimal angle difference we can measure.

We can reach

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

# Spin squeezing in an ensemble of atoms via interaction with light

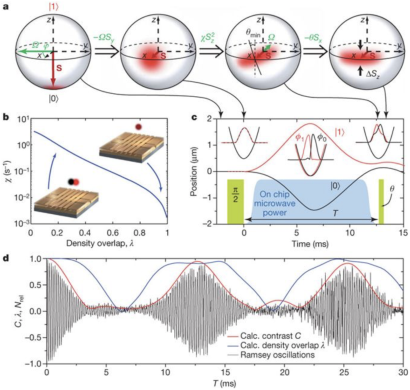



$10^{12}$ atoms, room temperature.

Julsgaard, Kozhekin, Polzik, Nature 2001.

# Spin squeezing in a Bose-Einstein Condensate via interaction between the particles

**Figure 1: Spin squeezing and entanglement through controlled interactions on an atom chip.**

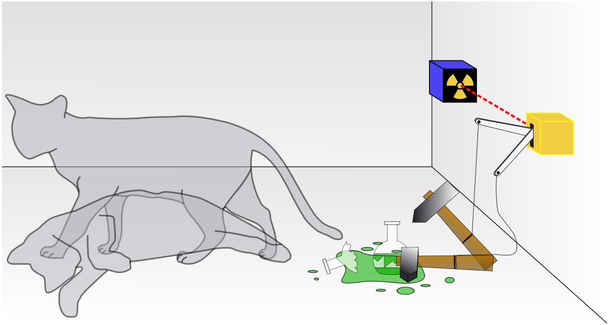



M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature 464, 1170-1173 (2010).

# Outline

# GHZ state=Schrödinger cat state

- A superposition of two macroscopically distinct states



# GHZ state

## Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000...00\rangle + |111...11\rangle).$$

- Superposition of all atoms in state "0" and all atoms in state "1".

# Metrology with the GHZ state

- Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000...00\rangle + |111...11\rangle),$$

- Unitary

$$|\Psi\rangle(\theta) = U_\theta|\mathrm{GHZ}_N\rangle, \qquad U_\theta = e^{-iJ_z\theta}.$$

- Dynamics

$$|\Psi\rangle(\theta) = \tfrac{1}{\sqrt{2}}(|000...00\rangle + e^{-iN\theta}|111...11\rangle),$$

# Metrology with the GHZ state II

- We measure

$$M = \sigma_x^{\otimes N},$$

which is the parity in the $x$-basis.

- Expectation value and variance

$$\langle M \rangle = \cos(N\theta), \qquad (\Delta M)^2 = \sin^2(N\theta).$$

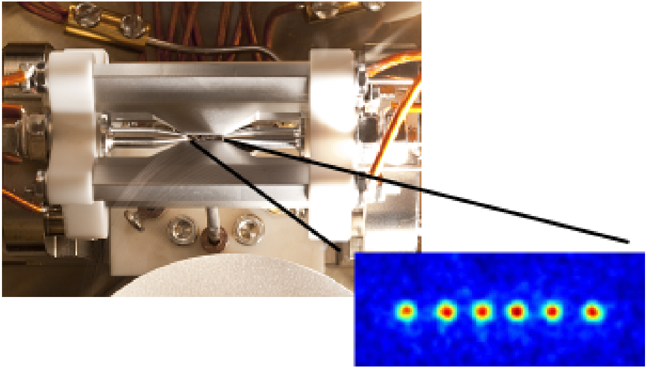
- For $\theta \approx 0$, the precision is

$$(\Delta\theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu N^2}.$$

[ e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);

ions: C. Sackett et *al.*, Nature 404, 256 (2000). ]

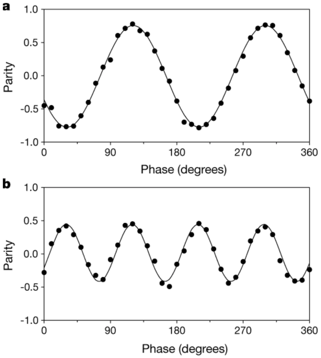
# Metrology with the GHZ state III



Quantum Computation with Trapped Ions, Innsbruck

# Metrology with the GHZ state IV

**Figure 2: Determination of $\rho_{(\uparrow\downarrow)}$.**




**a**, Interference signal for two ions; **b**, four ions. After the entanglement operation of Fig. 1, an analysis pulse with relative phase φ is applied on the single-ion $|\downarrow\rangle \leftrightarrow |\uparrow\rangle$ transition. As φ is varied, the parity of the *N* ions oscillates as cos *N*φ, and the amplitude of the oscillation is twice the magnitude of the density-matrix element $\rho_{(\uparrow\downarrow)}$. Each data point represents an average of 1,000 experiments, corresponding to a total integration time of roughly 10 s for each graph.

For four ions the curve oscillates faster than for two ions.

[ ions: C. Sackett et *al.*, Nature 404, 256 (2000). ]

# Heisenberg limit vs. shot-noise limit I

## Heisenberg limit vs. shot-noise limit

We reached the Heisenberg-limit

$$(\Delta\theta)^2 = \frac{1}{\nu N^2}.$$

The fully polarized state reached only the shot-noise limit

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

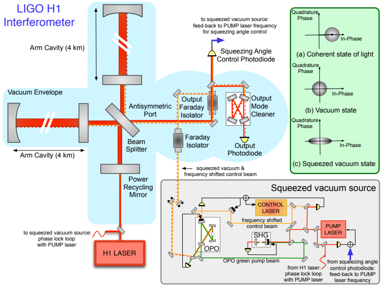
- "In electronics shot-noise originates from the discrete nature of electric charge." (Wikipedia)

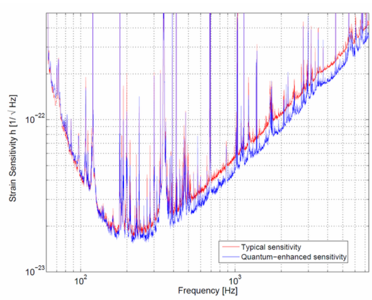
- For the fully polarized state, we use each particle individually for metrology, and then we average the results.

# Outline

# LIGO gravitational wave detector

The performance was enhanced with squeezed light.




The role of clock arm is played by the squeezed coherent state.

[ J. Aasi et al., Nature Photonics 2013. ]

# Outline

# Entanglement

**A state is (fully) separable if it can be written as**

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

# $k$-producibility/$k$-entanglement

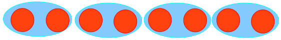
**A pure state is $k$-producible if it can be written as**

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

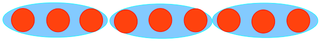
where $|\Phi_l\rangle$ are states of at most $k$ qubits.

A mixed state is $k$-producible, if it is a mixture of $k$-producible pure states.
[ e.g., Gühne, GT, NJP 2005. ]

- If a state is not $k$-producible, then it is at least $(k+1)$-particle entangled.

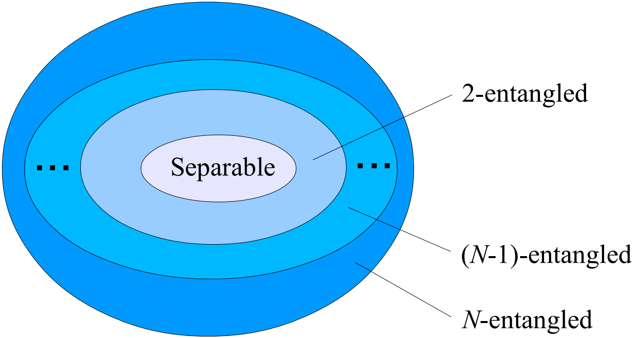


2-entangled



3-entangled

# $k$-producibility/$k$-entanglement II




$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

# Outline

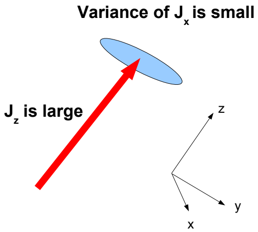
# The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

$$\xi_s^2 = N\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2}.$$

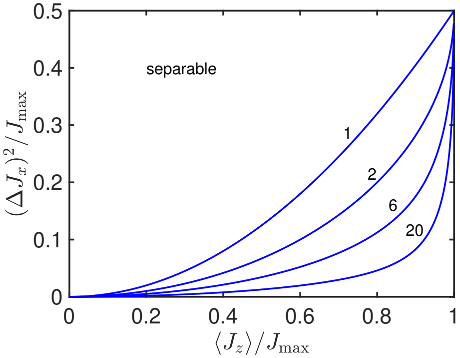
If $\xi_s^2 < 1$ then the state is entangled.
[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:



# Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").




- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

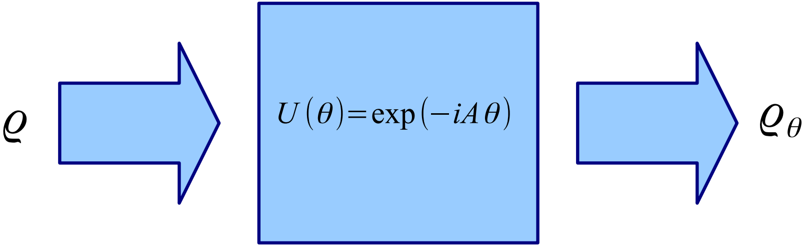
# Our experience so far

- We find that more spin squeezing/better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.

# Outline

# Quantum metrology

- Fundamental task in metrology




- We have to estimate $\theta$ in the dynamics

$$U = \exp(-iA\theta).$$

# The quantum Fisher information

## Cramér-Rao bound on the precision of parameter estimation

For the variance of the parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]}$$

holds, where $\nu$ is the number of repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The bound includes any estimation strategy, even POVM's.
- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

# Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

$$F_Q[|\Psi\rangle, A] = 4(\Delta A)^2{}_{\Psi}.$$

- For mixed states, it is convex

$$F_Q[\varrho, A] \leq \sum_k p_k F_Q[|\Psi_k\rangle, A],$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

# Quantum Fisher information - Some basic facts

- The larger the quantum Fisher information, the larger the achievable precision.
- For the totally mixed state it is zero for any $A$

$$F_Q[\varrho_{\rm cm}, A] = 0,$$

where $\varrho_{\rm cm} = \mathbb{1}/d$ is the completely mixed state and $d$ is the dimension.

- This is logical: the completely mixed states does not change under any Hamiltonian.
- For any state $\varrho$ that commutes with $A$, i.e., $\varrho A - A\varrho = 0$ we have

$$F_Q[\varrho, A] = 0.$$

# Outline

# Magnetometry with a linear interferometer

- The Hamiltonian $A$ is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

  There are no interaction terms.

- The dynamics rotates all spins in the same way.

# Quantum Fisher information for separable states

- Let us consider a pure product state of $N$ qubits

$$|\Psi\rangle_{\rm prod} = |\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes ... \otimes |\Psi^{(N)}\rangle.$$

- Since this is a pure state, we have $F_Q[\varrho, J_l] = 4(\Delta J_l)^2{}_{|\Psi\rangle_{\rm prod}}$.

- Then, for the product state we have

$$(\Delta J_l)^2{}_{|\Psi\rangle_{\rm prod}} = \sum_{n=1}^{N} (\Delta j_l^{(n)})^2{}_{|\Psi^{(n)}\rangle} \leq N \times \frac{1}{4},$$

where we used that for qubits $(\Delta j_l^{(n)})^2 \leq 1/4$.

- Since the quantum Fisher information is convex in the state, the bound is also valid for a mixture of product states, i.e., separable states

$$F_Q[\varrho, J_l] \leq N.$$

# The quantum Fisher information vs. entanglement

- For separable states of $N$ spin-1/2 particles (qubits)

$$F_Q[\varrho, J_l] \le N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most $k$-qubit entanglement ($k$ is divisor of $N$)

$$F_Q[\varrho, J_l] \le kN.$$

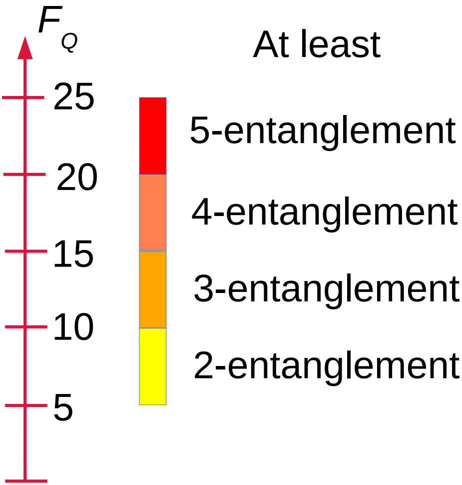
[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states of $N$ qubits

$$F_Q[\varrho, J_l] \le N^2.$$

# The quantum Fisher information vs. entanglement

5 spin-1/2 particles



# Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N}, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most $k$-particle entanglement ($k$ is divisor of $N$)

$$(\Delta\theta)^2 \geq \frac{1}{\nu k N}.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N^2}.$$

# Outline

# Noisy metrology: Simple example

- A particle with a state $\varrho_1$ passes trough a map that turns its internal state to the fully mixed state with some probability $p$ as

$$\epsilon_p(\varrho_1) = (1-p)\varrho_1 + p\frac{\mathbb{1}}{2}.$$

- This map acts in parallel on all the $N$ particles.

- Metrology with a spin squeezed state

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta J_x)^2}{\langle J_z\rangle^2} \geq \frac{1}{\nu}\frac{\frac{pN}{4}}{\frac{N^2}{4}} = p\frac{1}{\nu N} \propto \frac{1}{\nu N}.$$

- Shot-noise scaling if $p > 0$.

[G. Toth, and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).]

# Noisy metrology: General treatment

- In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

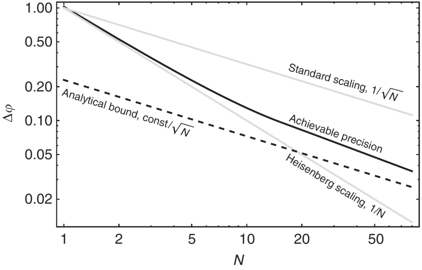



Figure from
[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

- Correlated noise is different.

# Take home message

- Quantum physics makes it possible to obtain bounds for precision of the parameter estimation in realistic many-particle quantum systems.
- Shot-noise limit: Non-entangled states lead to $(\Delta\theta)^2 \geq \frac{1}{\nu N}$.
- Heisenberg limit: Fully entangled states can lead to $(\Delta\theta)^2 = \frac{1}{\nu N^2}$.
- At the end, noise plays a central role.

# Reviews


- M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. 7, 125 (2009).
- V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics 5, 222 (2011).
- C. Gross, Spin squeezing, entanglement and quantum metrology with Bose-Einstein condensates, J. Phys. B: At.,Mol. Opt. Phys. 45, 103001 (2012).
- R. Demkowicz-Dobrzanski, M. Jarzyna, and J. Kolodynski, Chapter four-quantum limits in optical interferometry, Prog. Opt. 60, 345 (2015).
- L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: fundamentals and applications in quantum metrology, Rev. Mod. Phys. 90, 035005 (2018).

# Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,

Quantum metrology from a quantum information science perspective,

J. Phys. A: Math. Theor. 47, 424006 (2014),
special issue "50 years of Bell's theorem"
(open access).