Detecting metrologically useful multiparticle entanglement of Dicke states

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Outline

Why multipartite entanglement is important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

Outline

1. **Motivation**
   - Why multipartite entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Spin squeezing for Dicke states**
   - Entanglement detection close to Dicke states
   - Detection of multipartite entanglement close to Dicke states
   - Our conditions are stronger than the original conditions
4. **Detecting metrologically useful entanglement**
   - Basics of quantum metrology
   - Metrology with measuring $\langle J_z^2 \rangle$
   - Metrology with measuring any operator
5. **Scaling arguments**

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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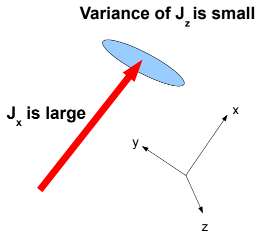
The standard spin-squeezing criterion

The spin squeezing criteria for entanglement detection is

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

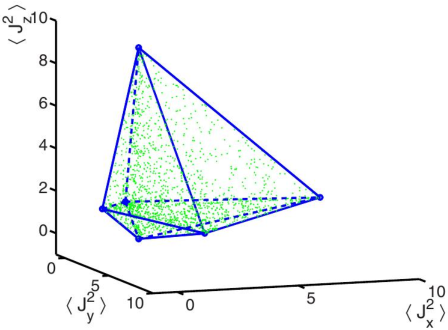
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \le \tfrac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge \tfrac{N}{2}, \qquad \text{(singlet)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \le (N-1)(\Delta J_m)^2 + \tfrac{N}{2}, \qquad \text{(Dicke state)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \ge \langle J_m^2 \rangle + \tfrac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]
[Singlets: Behbood *et al.*, Phys. Rev. Lett. 2014; GT, Mitchell, New. J. Phys. 2010.]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{2\mathrm{p}}$.
- Still, as we will see, we can even detect multipartite entanglement!

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Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011; GT, PRA 2012; GT and Apellaniz, JPHYSA, 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \tfrac{N}{2} \le (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$
$$\langle J_z^2 \rangle = 0.$$

Spin Squeezing Inequality for Dicke states II

Based on the above inequality, we define a new spin squeezing parameter

$$\xi_{\rm os}^2 = \frac{RHS}{LHS} = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2\rangle - \frac{N}{2}}.$$

[Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014)]

- For our Dicke state, the numerator is minimal, the denominator is maximal, $\xi_{\rm os}^2 = 0$. Dicke states are not detected by $\xi_{\rm s}^2$.
- For fully poalrized states $\xi_{\rm s}^2 \approx \xi_{\rm os}^2$.

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: C. Gross *et al.*, Nature 464, 1165 (2010).]

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

  and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2}}{J_{\max}}\right).$$

- Combine it with

$$\langle J_x^2 + J_y^2\rangle \leqslant J_{\max}(\tfrac{k}{2}+1) + \langle J_x\rangle^2 + \langle J_y\rangle^2,$$

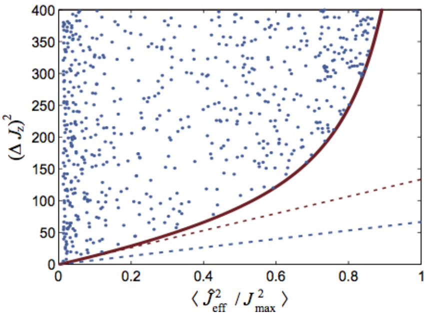
which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x^2 + J_y^2\rangle - J_{\max}(\frac{k}{2}+1)}}{J_{\max}}\right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Concrete example



- $N = 8000$ particles, and $J_{\text{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- Red dashed line: tangent of our curve.

Outline

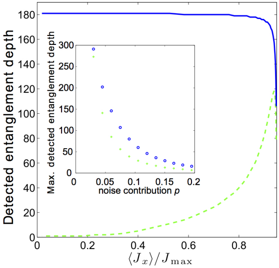
Our condition is stronger

- Consider spin squeezed states as ground states of

$$H(\Lambda) = J_z^2 - \Lambda J_x.$$

For $\Lambda = \infty$, the ground state is fully polarized. For $\Lambda = 0$, it is the symmetric Dicke state.

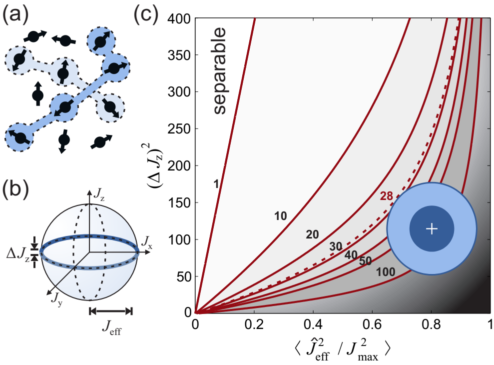
- Our condition vs. original condition for N=4000 and p=0.05



[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

Experimental results

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.

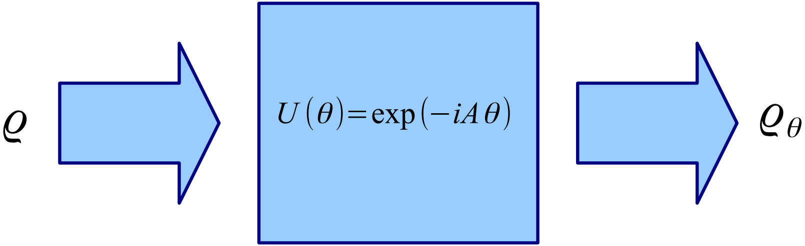


[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

  where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Outline

Metrology with Dicke states

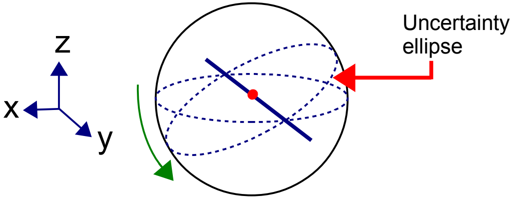
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)

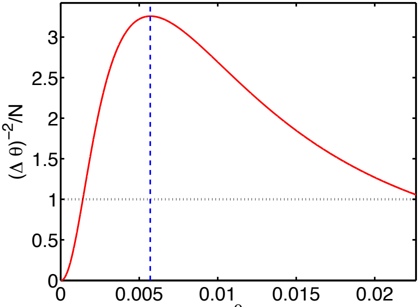


Metrology with Dicke states II

We measure $\langle J_z^2 \rangle$ to estimate θ. The precision is given by the error-propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta J_z^2)^2}{|\partial_\theta \langle J_z^2 \rangle|^2}.$$

- Precision as a function of θ for some noisy Dicke state



Formula for maximal precision

Parameter value for the maximum

$$\tan^2 \theta_{\text{opt}} = \sqrt{\frac{(\Delta J_z^2)^2}{(\Delta J_x^2)^2}}.$$

Consistency check: for the noiseless Dicke state we have $(\Delta J_z^2)^2 = 0$, hence $\theta_{\text{opt}} = 0$.

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision II

Maximal precision with a closed formula

$$(\Delta\theta)^2_{\rm opt} = \frac{2\sqrt{(\Delta J_z^2)^2(\Delta J_x^2)^2} + 4\langle J_x^2\rangle - 3\langle J_y^2\rangle - 2\langle J_z^2\rangle(1+\langle J_x^2\rangle) + 6\langle J_z J_x^2 J_z\rangle}{4(\langle J_x^2\rangle - \langle J_z^2\rangle)^2}.$$

- Given in terms of collective observables, like spin-squeezing criteria.
- Metrological usefulness can be verified without carrying out the metrological task.

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision III

- Some things are difficult to measure, they can be bounded

$$\langle J_z J_x^2 J_z\rangle = \frac{\langle J_z(J_x^2+J_y^2)J_z\rangle}{2} = \frac{\langle J_z(J_x^2+J_y^2+J_z^2)J_z\rangle - \langle J_z^4\rangle}{2} \leq \frac{N(N+2)}{8}\langle J_z^2\rangle - \frac{1}{2}\langle J_z^4\rangle.$$

- Equality holds for symmetric states.

[I. Apellaniz, B. Lücke, J. Peise, C. Klempt, GT. arXiv:1412.3426.]

Experimental test of our formula

- Trying the bound for the experimental data for $N = 7900$ particles

$$\langle J_z^2 \rangle = 112 \pm 31, \qquad \langle J_z^4 \rangle = 40 \times 10^3 \pm 22 \times 10^3,$$
$$\langle J_x^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6, \quad \langle J_x^4 \rangle = 6.2 \times 10^{13} \pm 0.8 \times 10^{13}.$$

- Hence, we obtain

$$\frac{(\Delta\theta)^{-2}_{\text{opt}}}{N} \geq 3.7 \pm 1.5.$$

- Remember, for states for at most k-particle entanglement we have

$$(\Delta\theta)^{-2} \leq F_Q[\varrho, J_l] \leq kN.$$

- Thus, four-particle entanglement is detected for this particular measurement.

Comparison with the quantum Fisher information

- For the noiseless Dicke state, the optimal operator to measure is

$$M = J_z^2.$$

- For a noisy Dicke state, this is not true any more. In this case it can happen that

$$(\Delta\theta)^{-2} = \frac{|\partial_\theta \langle J_z^2 \rangle|^2}{(\Delta J_z^2)^2} \ll F_Q[\varrho, J_y].$$

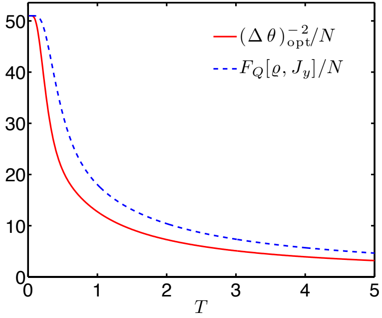
- We should estimate the quantum Fisher information.

Comparison with the quantum Fisher information II

- Noisy states

$$\varrho_{\mathrm{th}}(T) \propto \sum_{m=0}^{N} e^{-\frac{(m-N/2)^2}{T}} |\mathrm{D}_N^{(m)}\rangle\langle\mathrm{D}_N^{(m)}|,$$

- Here $T = 0$ perfect symmetric Dicke state, $T > 0$ noisy state. $N = 100$ particles



Outline

Bounding the qFi based on collective measurements

Bound for the quantum Fisher information for spin squeezed states

$$F[\varrho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

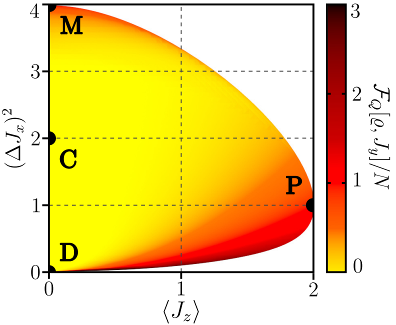
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Bounding the qFi based on collective measurements II

- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles



P=fully polarized state, D=Dicke state, C=completely mixed state, M=mixture of $|00..000\rangle_x$ and $|11..111\rangle_x$

[Apellaniz *et al.*, in preparation.]

Next step

- Estimating the quantum Fisher information close to Dicke states.

[Apellaniz *et al.*, in preparation.]

Arguments about scaling

For large N, the effects of noise lead to shot-noise scaling.

[R. Demkowicz-Dobrzanski, J. Kolodynski, M. Guta, Nat. Commun. 3, 1063 (2012); B.M. Escher, R.L. de Matos Filho, L. Davidovich, Nat. Phys. 7 (5), 406-411 (2011).]

- Would still be possible to obtain a scaling better than $F_Q \propto O(N)$?

We should make more Dicke states

- Thus, Dicke states and GHZ states might be difficult to prepare due to the findings above. (Maybe, BEC is different?)
- Experiments with such states are needed to understand whether macroscopic superpositions can be prepared.
- Dicke states are not very sensitive to particle loss, GHZ states are very sensitive.
- Dicke states can be detected by collective observables, GHZ states need full N-nody correlations.
- There are not such limitations for W-states and cluster states, however, they are not useful for metrology.

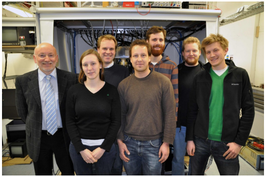
Scaling is not all

- Shot-noise scaling for large *N* is not all the story.

How large is “large” N?

- Example: the probability of correct operation of a chip decreases exponentially with the surface, still we have quite large digital circuits.

Project participants



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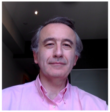

I.L. Egusquiza

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O. Gühne

Siegen

Summary

- Detection of multipartite entanglement and metrological usefulness close to Dicke states, by measuring collective quantities only.

Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014).

Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt,
PRL 112, 155304 (2014)
(synopsis at physics.aps.org);

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015).

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