Activating hidden metrological usefulness

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Photos

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Outline

1. **Motivation**
   - What are entangled states useful for?
2. **Bacground**
   - Quantum Fisher information
   - Error propagation formula
3. **Metrological gain and the optimal local Hamiltonian**
   - Metrological usefulness of a quantum state.
   - Activation of metrological usefulness
   - Optimal local Hamiltonian
   - Bipartite pure entangled states

What are entangled states useful for?

- Entangled is needed for beating the shot-noise limit in quantum metrology.

- However, not all entangled states are more useful than separable states.
  [P. Hyllus, O. Gühne, A. Smerzi, PRA 2010.]

- Intriguing questions:

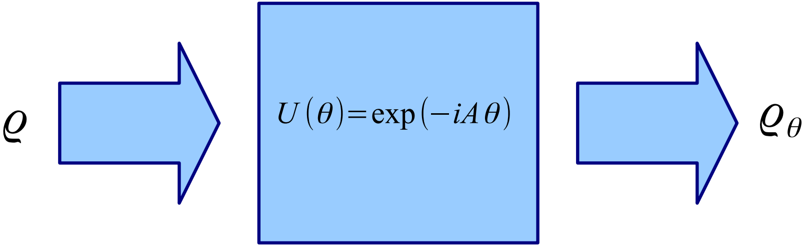
  - Can a quantum state become useful metrologically, if an ancilla or a second copy is added?

  - How to find the local Hamiltonian, for which a quantum state is the most useful compared to separable states?

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where where m is the number of independent repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

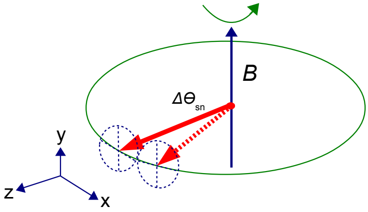
where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

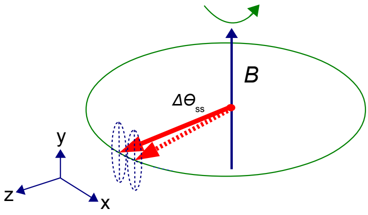
Special case $A = J_l$

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer





The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

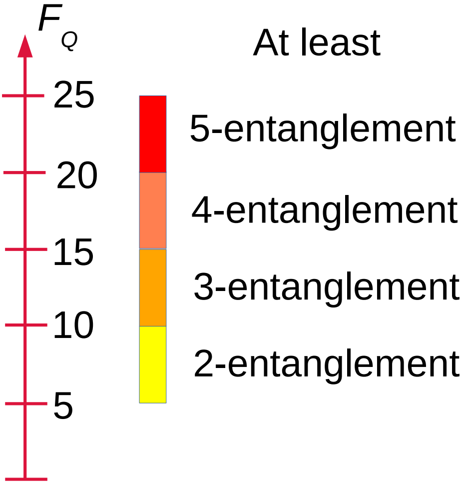
- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

The quantum Fisher information vs. entanglement II

5 spin-1/2 particles



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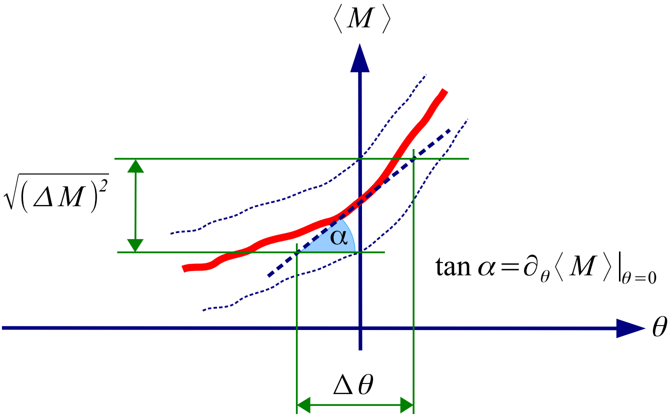
Error propagation formula

- Measure an operator M to get the estimate θ. The error propagation formula is

$$(\Delta\theta)^2_M = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



Relation between $(\Delta\theta)^2$ and the error propagation formula $(\Delta\theta)^2_M$

- The relation

$$(\Delta\theta)^2 \geq \frac{1}{m}(\Delta\theta)^2{}_{M_{\rm opt}}$$

  holds, where m is the number of independent repetitions and $M_{\rm opt}$ is the optimal observable.

- The relation can be saturated if m is large and the distribution fulfills certain requirements.

  [L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Rev. Mod. Phys. 2018.]

- Moreover,

$$(\Delta\theta)^2{}_M \geq (\Delta\theta)^2{}_{M_{\rm opt}} = \frac{1}{F_Q[\varrho, A]}.$$

  [M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015. For a summary, see, e.g., the Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

Outline

Metrological usefulness

- Metrological gain for a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

  where $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})$ is the maximum of the QFI for separable states.

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\mathrm{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}.$$

- A state ϱ is useful if $g(\varrho) > 1$.

- The metrological gain is convex in the state.
  [G. Toth, T. Vertesi, P. Hordecki, R. Horodecki, PRL 2020.]

- We would like to detmine g.

Metrological usefulness II

- So we would like optimize over local $\mathcal{H}$ the expression

$$g(\varrho) = \max_{\text{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})}.$$

- First observation: we really optimize the QFI over $\mathcal{H}$, but we normalize it with something meaningful.
- This is needed, since otherwise $\mathcal{H}' = 100\mathcal{H}$ would be better than $\mathcal{H}$.
- Second observation: difficult task, since both the numerator and the denominator depend on $\mathcal{H}$.

Metrological usefulness III

- The local Hamiltonians can be given as

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2.$$

- The separable limit is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\max}(\mathcal{H}_n) - \sigma_{\min}(\mathcal{H}_n)]^2.$$

[M. A. Ciampini, N. Spagnolo, C. Vitelli, L. Pezze, A. Smerzi, and F. Sciarrino, Sci. Rep. 2016; See also G. Tóth, Vértesi, Phys. Rev. Lett. 2018.]

Maximally entangled state

- Difficult to obtain $g(\varrho)$ and the optimal local Hamiltonian for any ϱ.
- As a first step, we consider the $d \times d$ maximally entangled state

$$|\Psi^{(\mathrm{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle|k\rangle.$$

- The optimal Hamiltonian is

$$\mathcal{H}^{(\mathrm{me})} = D \otimes \mathbb{1} + \mathbb{1} \otimes D,$$

where

$$D = \mathrm{diag}(+1, -1, +1, -1, ...).$$

Maximally entangled state II

The 3×3 noisy quantum state

$$\varrho_{AB}^{(p)} = (1-p)|\Psi^{(\mathrm{me})}\rangle\langle\Psi^{(\mathrm{me})}| + p\mathbb{1}/d^2,$$

is useful if

$$p < \frac{25-\sqrt{177}}{32} \approx 0.3655,$$

while for larger p's it is not useful.

- Note that it is entangled if

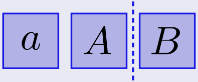
$$p < \frac{2}{3}.$$

Outline

Activation by an ancilla qubit

If a pure ancilla qubit is added

$$\varrho^{(\mathrm{anc})} = |0\rangle\langle 0|_a \otimes \varrho_{AB}^{(p)}.$$



then the state is useful if

$$p < 0.3752.$$

(For a single copy, the limit was $p < 0.3655$.)

- The Hamiltonian is

$$\mathcal{H}^{(\mathrm{anc})} = 1.2 C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B,$$

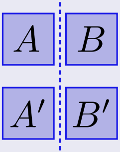
where

$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_a + \mathbb{1}_a \otimes (|2\rangle\langle 2|_a - |1\rangle\langle 1|_a).$$

Activation by a second copy

If a second copy is added

$$\varrho^{(\mathrm{tc})} = \varrho_{AB}^{(p)} \otimes \varrho_{A'B'}^{(p)}.$$



then the state is useful if

$$p < 0.4164.$$

(For a single copy, the limit was $p < 0.3655$.)

- The Hamiltonian is

$$\mathcal{H}^{(\mathrm{tc})} = D_A \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}.$$

Outline

Method for finding the optimal local Hamiltonian - Exisiting work for qubits

- For qubits, the local Hamiltonians with eigenvalues $+1$ and -1 differ from each other by local unitaries

$$\mathcal{H} = U_1 \sigma_z U_1^\dagger \otimes \mathbb{1} + \mathbb{1} \otimes U_2 \sigma_z U_2^\dagger.$$

- It is possible to obtain upper bounds on the quantum Fisher information.

- All pure two-qubit entangled states are useful, while not all pure multi-qubit entangled states are useful.

  [P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).]

- When looking at $\mathcal{F}_Q/\mathcal{F}_Q^{(\rm sep)}$, the value of $\mathcal{F}_Q^{(\rm sep)}$ does not depend on the particular Hamiltonian. For instance for spin operators $\mathcal{F}_Q^{(\rm sep)} = N$.

  [L. Pezze and A. Smerzi, Phys. Rev. Lett. 2009.]

Method for finding the optimal local Hamiltonian

- The case of qudits is more complicated than the case of qubits, since the local Hamiltonians cannot be converted to each other by unitaries.

- Direct maximization of $\mathcal{F}_Q[\varrho,\mathcal{H}]$ over $\mathcal{H}$ is difficult: it is convex in $\mathcal{H}$.

- Let us consider the error propagation formula

$$(\Delta\theta)^2{}_M = \frac{(\Delta M)^2}{|\partial_\theta \langle M\rangle|^2} \equiv \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2},$$

  which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\varrho,\mathcal{H}] \geq 1/(\Delta\theta)^2{}_M.$$

  [M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015. For a summary, see, e.g., the Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

Method for finding the optimal Hamiltonian II

We compute the QFI as

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = \max_M \frac{\langle i[M, \mathcal{H}] \rangle_\varrho^2}{(\Delta M)^2}.$$

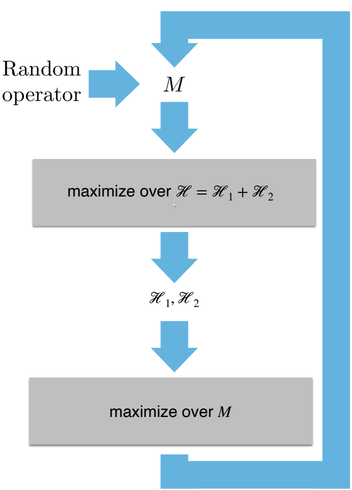
The maximum of the QFI over local Hamiltonians can be obtained as

$$\max_{local\ \mathcal{H}} \mathcal{F}_Q[\varrho, \mathcal{H}] = \max_{local\ \mathcal{H}} \max_M \frac{\langle i[M, \mathcal{H}] \rangle_\varrho^2}{(\Delta M)^2}.$$

Similar idea for optimizing over the state, rather than over $\mathcal{H}$:

[K. Macieszczak, arXiv:1312.1356; K. Macieszczak, M. Fraas, and R. Demkowicz-Dobrzański, New J. Phys. 16, 113002 (2014); Tóth and Vértesi, Phys. Rev. Lett. (2018).]

See-saw algorithm



The precision cannot get worse with the iteration!

Note that $\mathcal{H}_1, \mathcal{H}_2$ fulfill

$$c_n \mathbb{1} \pm \mathcal{H}_n \geq 0.$$

Numerical results

- We remember that the 3×3 isotropic state is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

- Then, we have the following results for activation.

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

Outline

Single copy of pure states

All entangled bipartite pure states are metrologically useful.

- *Proof.*—For the two-qubit case, see
  P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).

- General case, pure state with a Schmidt decomposition

$$|\Psi\rangle = \sum_{k=1}^{s} \sigma_k |k\rangle_A |k\rangle_B,$$

  where s is the Schmidt number, and the real positive σ_k Schmidt coefficients are in a descending order.

- We define

$$\mathcal{H}_A = \sum_{n=1,3,5,\ldots,\tilde{s}-1} |+\rangle\langle+|_{A,n,n+1} - |-\rangle\langle-|_{A,n,n+1},$$

  where $\tilde{s}$ is the largest even number for which $\tilde{s} \leq s$, and

$$|\pm\rangle_{A,n,n+1} = (|n\rangle_A \pm |n+1\rangle_A)/\sqrt{2}.$$

Single copy of pure states II

- We define $\mathcal{H}_B$ in a similar manner.

- We also define the collective Hamiltonian

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_B.$$

  Then, we have $\langle \mathcal{H}_{AB} \rangle_\Psi = 0$.

- Direct calculation yields

$$\mathcal{F}_Q[|\Psi\rangle, \mathcal{H}_{AB}] = 4(\Delta\mathcal{H}_{AB})^2{}_\Psi = 8 \sum_{n=1,3,5,\ldots,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2,$$

  which is larger than the separable bound, $\mathcal{F}_Q^{(\mathrm{sep})} = 8$, whenever the Schmidt rank is larger than 1. □

Infinite number of copies

In the infinite copy limit, all bipartite pure entangled states are maximally useful.

[Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

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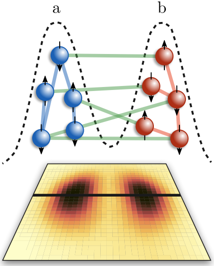
G. Vitagliano, M. Fadel, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt, G. Tóth, *Detecting Einstein-Podolsky-Rosen steering and bipartite entanglement in split Dicke states*, arXiv:2104.05663.

- An entanglement criterion somewhat stronger and simpler than in K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Science 360, 416 (2018).
- A simple EPR-steering criterion.



Summary

- Some entangled quantum states that are not useful metrologically, can still be made useful, if an ancilla or an additional copy is added.
- We have shown a general method to get the optimal local Hamiltonian for a quantum state.
- All bipartite pure entangled states are useful metrologically. If infinite number of copies are given, they are maximally useful.

See:

Géza Tóth, Tamás Vértesi, Paweł Horodecki, Ryszard Horodecki,

Activating hidden metrological usefulness,

Phys. Rev. Lett. 125, 020402 (2020). (open access)

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