Activation of metrologically useful genuine multipartite entanglement, arXiv:2203.05538

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Outline

What are entangled states useful for?

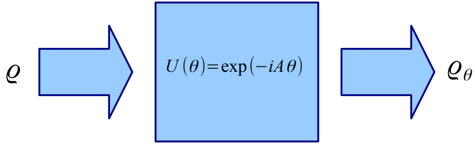
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- However, not all entangled states are more useful than separable states.
- Intriguing questions:
  - Can we activate the metrological usefulness of quantum states, if we use several copies?
  - Does it work for large systems? Can it be practical?

Outline

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho,A]},$$



where where m is the number of independent repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states of N qubits

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

L. Pezze, A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009);
P. Hyllus, O. Gühne, A. Smerzi, Phys. Rev. A 82, 012337 (2010)

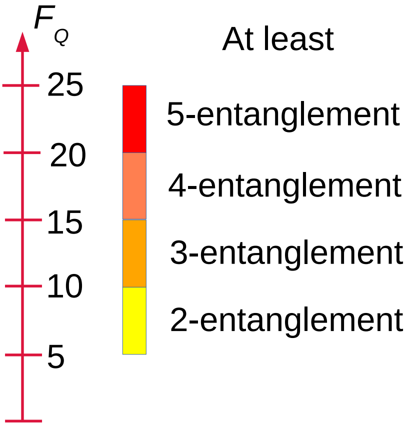
- For states with at most k-particle entanglement (tight bound if k is a divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012);
GT, Phys. Rev. A 85, 022322 (2012).

The quantum Fisher information vs. entanglement

5 spin-1/2 particles



(For simplicity, we used $F_Q[\varrho, J_l] \leq kN$, which is not tight.)

Metrological usefulness

- Qudits are more complicated!
- Metrological gain for a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

  where $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})$ is the maximum of the QFI for separable states.

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\mathrm{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}.$$

- A state ϱ is useful if $g(\varrho) > 1$.
- The metrological gain is convex in the state.

  G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

Metrologically useful k-entanglement

- k-particle entanglement means that we could not make trivially the experiment from $(k-1)$-particle experiments.
- The state is not a mixture of product states

$$\varrho_1 \otimes \varrho_2 \otimes \varrho_3 \otimes \dots$$

  such that all ϱ_l has at most $(k-1)$ qubits.
- If $g > k-1$ then we have metrologically useful k-particle entanglement, that is, additionally, the state is more useful than any states mentioned above.

Metrologically useful genuine multipartite entanglement

- If $g > N - 1$ then the state possesses metrologically useful genuine multipartite entanglement.

- On the one hand, the state has genuine multipartite entanglement. Thus, the experiment cannot be "put together" from smaller experiments in a trivial way.

- Such entanglement is the target of many experiments in photons, ions and cold gases.

  A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera,
  Classification of Mixed Three-Qubit States, PRL 2001;
  M. Bourennane M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter,
  O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, PRL 2004.

- On the other hand, the state is also metrologically better than any state put together from smaller experiments in a trivial way.

Outline

1. Motivation
   - What are entangled states useful for?

2. Metrological gain and the optimal local Hamiltonian
   - Metrological usefulness of a quantum state
   - Example for activation in small systems
   - Activation in the many-particle case

Method for maximizing g

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\text{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})} \begin{array}{l} \leftarrow \text{metrological performance of } \varrho \\ \leftarrow \text{best metrological performance of} \\ \quad \text{separable states} \end{array}$$

- It is a fundamental quantity in metrology!

- Difficult to compute, since $\mathcal{H}$ is in both the numerator and the denominator!

- We reduce the problem to maximize $\mathcal{F}_Q$ over a set of local Hamiltonians.

Method for finding the optimal local Hamiltonian I

- Direct maximization of $\mathcal{F}_Q[\varrho,\mathcal{H}]$ over $\mathcal{H}$ is difficult: it is convex in $\mathcal{H}$.

- Let us consider the error propagation formula

$$(\Delta\theta)^2{}_M = \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2},$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\varrho,\mathcal{H}] \geq 1/(\Delta\theta)^2{}_M.$$

M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; K. Macieszczak, arXiv:1312.1356; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015. For a summary, see, e.g., the Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.

Method for finding the optimal Hamiltonian II

The maximum over local Hamiltonians can be obtained as

$$\max_{\text{local } \mathcal{H}} \mathcal{F}_Q[\varrho, \mathcal{H}] = \max_{\text{local } \mathcal{H}} \max_{M} \frac{\langle i[M, \mathcal{H}] \rangle_{\varrho}^2}{(\Delta M)^2}.$$

G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

See-saw has been used for optimizing over the state, rather than over $\mathcal{H}$:

K. Macieszczak, arXiv:1312.1356; K. Macieszczak, M. Fraas, and R. Demkowicz-Dobrzański, New J. Phys. 16, 113002 (2014);
Tóth and Vértesi, Phys. Rev. Lett. (2018).

Example: Maximally entangled state

- We consider the $d \times d$ maximally entangled state

$$|\Psi^{(\mathrm{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle|k\rangle.$$

- The optimal Hamiltonian is

$$\mathcal{H}^{(\mathrm{me})} = D \otimes \mathbb{1} + \mathbb{1} \otimes D,$$

where

$$D = \mathrm{diag}(+1, -1, +1, -1, ...).$$

- We add white noise.

Numerical results

- The 3×3 isotropic state is useful if for the noise

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

- Then, we have the following results for two copies.

	Analytic example	Numerics
Second copy	0.4164	0.4170

- In the case of two copies,
  the metrological usefulness has been activated in the spirit of P. Horodecki, M. Horodecki and R. Horodecki, PRL 1989!

G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

Outline

1. **Motivation**
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2. **Metrological gain and the optimal local Hamiltonian**
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Multicopy metrology without interaction

- M copies of a quantum state, all undergoing a dynamics governed by the Hamiltonian $\mathcal{H}$.

- For the quantum Fisher information we obtain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}^{\otimes M}] = M\mathcal{F}_Q[\varrho, \mathcal{H}],$$

while the maximum for separable states also increases

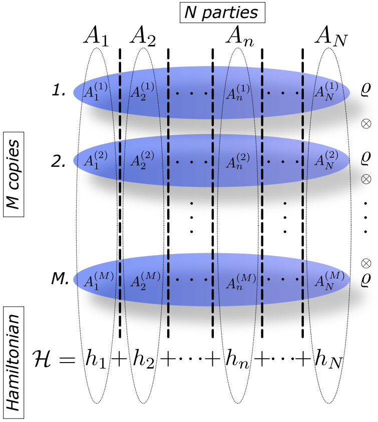
$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}^{\otimes M}) = M\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}).$$

- The metrological gain does not change

$$g_{\mathcal{H}^{\otimes M}}(\varrho^{\otimes M}) = g_{\mathcal{H}}(\varrho).$$

- (Unbiased estimators.)

Multicopy metrology with interaction



- Metrology with *M* copies of an *N*-partite quantum state ϱ.
- There is no interaction between particles corresponding to different parties.

Numerical results

- Based on numerics, the optimal local Hamiltonian turns out to be a sum of correlation terms

$$\mathcal{H} = h_1 + h_2 + ... + h_N,$$

where h_n are correlations

$$h_n = \otimes_{m=1}^{M} h_{A_n^{(m)}}.$$

- Remember, n^{th} qubit, m^{th} copy.

Multicopy metrology with interaction

Result 1.—Consider entangled states living in

$$\{|000..00\rangle, |111..11\rangle, ..., |d-1, d-1, .., d-1\rangle\}$$

subspace.

They are maximally useful in the limit of large number of copies

The maximally achievable metrological usefulness is attained *exponentially fast in the number of copies*.

Proof.—Consider

$$\varrho = \sum_{k,l=0}^{d-1} c_{kl}(|k\rangle\langle l|)^{\otimes N}, \quad \mathcal{H} = \sum_{n=1}^{N} (D^{\otimes M})_{A_n},$$

with $D = \mathrm{diag}(+1, -1, +1, -1, ...)$.

Multicopy metrology with interaction

- We use the mapping N qudits $\rightarrow$ 1 qudit

$$\varrho \rightarrow \tilde{\varrho} = \sum_{k,l=0}^{d-1} c_{kl}|k\rangle\langle l|, \quad \mathcal{H} \rightarrow \tilde{\mathcal{H}} = ND^{\otimes M},$$

  for which $\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = \mathcal{F}_Q[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}]$ holds.

- We can bound the quantum Fisher information as

$$\mathcal{F}_Q[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}] \geq 4I_{\tilde{\varrho}^{\otimes M}}(\tilde{\mathcal{H}}),$$

  where the Wigner-Yanase skew information is

$$I_{\tilde{\varrho}^{\otimes M}}(\tilde{\mathcal{H}}) = N^2[1 - \mathrm{Tr}(\sqrt{\tilde{\varrho}}D\sqrt{\tilde{\varrho}}D)^M].$$

- If $M \rightarrow \infty$, if $[\sqrt{\tilde{\varrho}}, D] \neq 0$ then the skew information above converges to the maximum. All such states are entangled.
- All other states are separable.

Multicopy metrology with interaction

- The Wigner-Yanase skew information can be written out as follows for $d = 2$

$$\frac{I}{N^2} = -\left[\frac{8c_{01}^2\sqrt{-c_{00}^2 + c_{00} - c_{01}^2} + 4(c_{00} - 1)c_{00} + 1}{(1 - 2c_{00})^2 + 4c_{01}^2}\right]^M$$

if $c_{01} \neq 0$, otherwise $I = 0$.

- Moreover, if $c_{00} = c_{11} = 1/2$ then this can be simplified to

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] \geq 4I(c_{01}, N) = N^2[1 - (1 - 4|c_{01}|^2)^{M/2}]. \quad \square$$

Multicopy metrology with interaction

Result 2.—For qubits, to achieve the maximal usefulness for the states the following operator has to be measured

$$\mathcal{M} = \sum_{m=1}^{M} Z^{\otimes(m-1)} \otimes Y \otimes Z^{\otimes(M-m)},$$

where we define the multi-qubit operators acting on a single copy

$$Y = \begin{cases} \sigma_y^{\otimes N} & \text{for odd } N, \\ \sigma_x \otimes \sigma_y^{\otimes(N-1)} & \text{for even } N, \end{cases}$$

$$Z = \sigma_z \otimes \mathbb{1}^{\otimes(N-1)}.$$

By taking sufficiently many copies, the precision corresponding to the metrologically useful GME ($g = N$) can be approached fast.

The required number of copies depends on how noisy the state is.

Multicopy metrology with interaction

- Consider the N-qubit state

$$\varrho(p,q,r) = p|\mathrm{GHZ}_q\rangle\langle\mathrm{GHZ}_q| + (1-p)[r(|0\rangle\langle 0|)^{\otimes N} + (1-r)(|1\rangle\langle 1|)^{\otimes N}].$$

  Here, $0 < p \leq 1$, $0 \leq r \leq 1$, and

$$|\mathrm{GHZ}_q\rangle = \sqrt{q}|000..00\rangle + \sqrt{1-q}|111..11\rangle,$$

  where $0 < q < 1$ is real.

- The error propagation formula:

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2}.$$

Multicopy metrology with interaction II

- We obtain

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1/[4q(1-q)] + (M-1)p^2}{4MN^2p^2}.$$

- If the condition

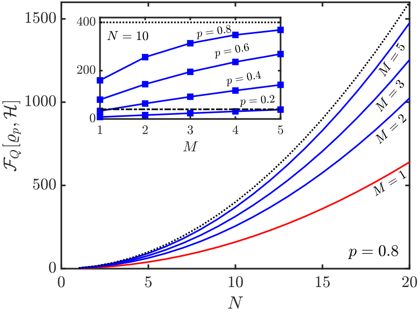
$$1/[4q(1-q)] \ll (M-1)p^2$$

is fulfilled and $M \gg 1$ holds, we have

$$(\Delta\theta)^2_{\mathcal{M}} \approx \frac{1}{4N^2}.$$

$\rightarrow$ Heisenberg limit, the best achievable precision.

Multicopy metrology with interaction



- Multicopy metrology with the noisy GHZ state for $p = 0.8$, with $p_{\text{noise}} = 1 - p = 0.2$ with $\varrho_{\text{noise}} = (|000\rangle\langle 000| + |111\rangle\langle 111|)/2$.
- (solid) The lower bound on the QFI depending on M.
- (dotted) The maximum of the quantum Fisher information, $4N^2$.

- (inset) QFI depending on M for $N = 10$ for various values for p.
- (dotted) The maximum of the QFI, $\mathcal{F}_Q^{(\text{max})} = 400$.
- (dashed dotted) The bound for separable states is $\mathcal{F}_Q^{(\text{sep})} = 40$.

Simple example

- Let us consider $M = 2$ copies of the 3-qubit state

$$\varrho_p = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-\mathrm{p})\frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|),$$

with $p = 0.8$.

- Then, we have

$$\mathcal{F}_Q[\varrho, H_2] = 28.0976, \qquad \text{(2 copies)}$$

while for $M = 1$ we have

$$\mathcal{F}_Q[\varrho, H_1] = 23.0400. \qquad \text{(1 copy)}$$

- In both cases,

$$\mathcal{F}_Q^{(\mathrm{sep})}(H_k) = 12,$$

hence for the metrological gain

$$g_1 = 1.92 < g_2 = 2.34.$$

Simple example II

- Considering the state

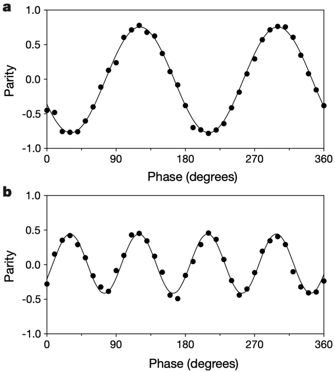
$$\varrho_p = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-\mathrm{p})\frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|),$$

  we took care of phase flip errors.

- We can also correct bitflip errors in the usual way, if the state is outside of the $\{|000\rangle, |111\rangle\}$ subspace.

Simple example III

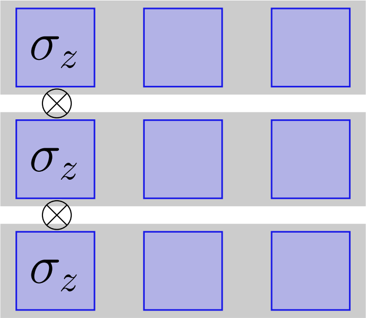
- Directly relevant to experiments with GHZ states!
- One can obtain maximal visibility.



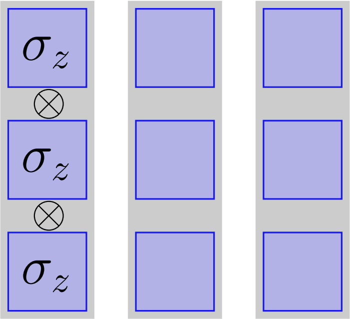
$N = 2$ and $N = 4$ particles, Sackett *et al.*, Experimental entanglement of four particles, Nature (2000).

Comparison to error correction

$M = 3$ copies, $N = 3$ qubits



3 logical qubits,
1 logical qubit=3 physical qubits



W. Dür, M. Skotiniotis, F. Fröwis, B. Kraus, Phys. Rev. Lett. (2014).

Comparison to error correction II

- How do we store a three-qubit GHZ state?

- Multicopy metrology:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$
$$H = \sigma_z^{(1)}\sigma_z^{(4)}\sigma_z^{(7)} + \sigma_z^{(2)}\sigma_z^{(5)}\sigma_z^{(8)} + \sigma_z^{(3)}\sigma_z^{(6)}\sigma_z^{(9)}.$$

Improves performance without syndrome measurements.

- Error correction for bit-flip code (phase-flip is similar):

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\,000\,000\rangle + |111\,111\,111\rangle),$$
$$H = \sigma_z^{(1)}\sigma_z^{(2)}\sigma_z^{(3)} + \sigma_z^{(4)}\sigma_z^{(5)}\sigma_z^{(6)} + \sigma_z^{(7)}\sigma_z^{(8)}\sigma_z^{(9)},$$

+ error syndrome measurements + error correction.

Summary

- We discussed metrology with several copies of the quantum state.
- We can obtain a state with metrological useful genuine multipartite entanglement for very weakly entangled states.

See: R. Trényi, Á. Lukács, P. Horodecki, R. Horodecki, T. Vértesi, and G. Tóth,

Activation of metrologically useful genuine multipartite entanglement, arXiv:2203.05538.

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