Quantum metrology (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

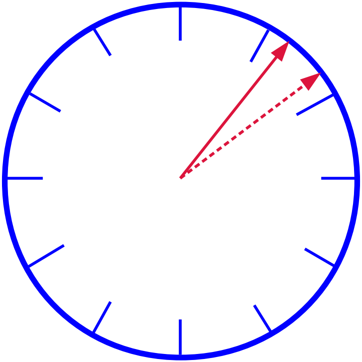
Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum systems, i.e., in cold gases, trapped cold ions or photons.
- Can such quantum systems outperform classical systems in something useful, i.e., metrology?

Outline

Classical case: Estimating the angle of a clock arm

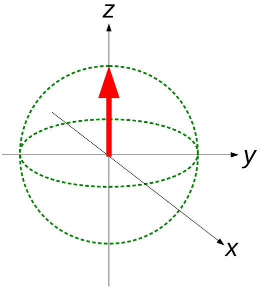
- Arbitrary precision ("in principle").



Outline

Quantum case: A single spin-$1/2$ particle

- Spin-1/2 particle polarized in the *z* direction.



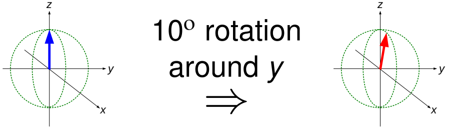
- We measure the spin components.

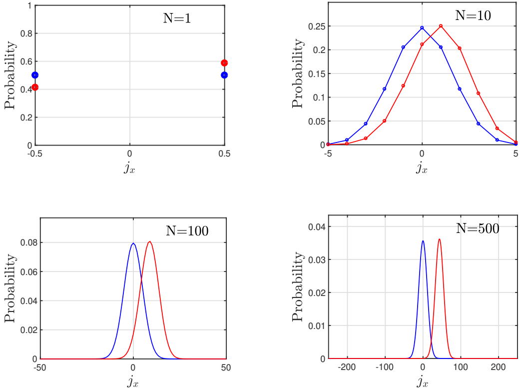
$$j_x \begin{array}{l} \nearrow \; +1/2, \;\; 50\% \\ \\ \searrow \; -1/2, \;\; 50\% \end{array} \qquad j_y \begin{array}{l} \nearrow \; +1/2, \;\; 50\% \\ \\ \searrow \; -1/2, \;\; 50\% \end{array} \qquad j_z \rightarrow +1/2$$

Quantum case: A single spin-$1/2$ particle II

- We cannot measure the three spin coordinates exactly j_x, j_y, j_z.
- In quantum physics, we can get only discrete outcomes in measurement. In this case, $+1/2$ and $-1/2$.
- A single spin-1/2 particle is not a good clock arm.

Several spin-1/2 particles

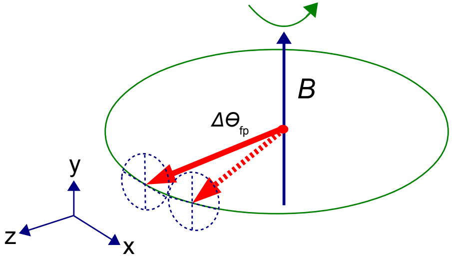




Outline

Magnetometry with the fully polarized state

- N spin-$1/2$ particles, all fully polarized in the z direction.
- Magtetic field B points to the y direction.



- Note the uncertainty ellipses. $\Delta\theta_{\mathrm{fp}}$ is the minimal angle difference we can measure.

Magnetometry with the fully polarized state II

- Collective angular momentum components

$$J_l := \sum_{n=1}^{N} j_l^{(n)}$$

for $l = x, y, z$, where $j_l^{(n)}$ are single particle operators.

- Dynamics

$$|\Psi\rangle = U_\theta |\Psi_0\rangle, \qquad U_\theta = e^{-iJ_y\theta},$$

where $\hbar = 1$.

- Rotation around the y-axis.

Magnetometry with the fully polarized state III

- Let us assume, that we have an $M(\Theta)$ function.
- We know that there is an ΔM error in M.
- How much is the error $\Delta\theta$ in θ?
- It is given by the classical error propagation formula:

$$\Delta\theta \approx \frac{\Delta M}{dM/d\theta}.$$

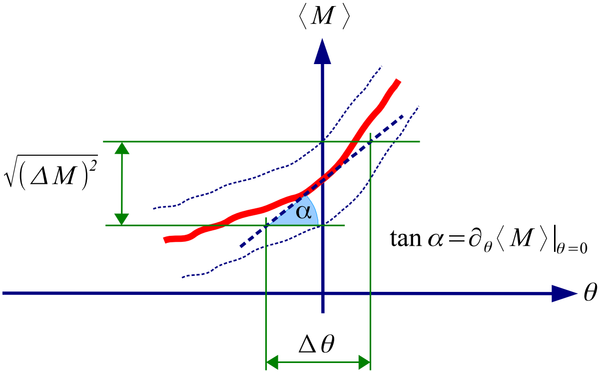
- It tells us how the error in M "propagates" to θ.

Magnetometry with the fully polarized state IV

- Measure an operator M to get the estimate θ.

- To obtain the precision of estimation, we can use the error propagation formula

$$(\Delta\theta)^2_M = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{(\Delta M)^2}{|\langle i[M,H]\rangle|^2}.$$



Magnetometry with the fully polarized state V

- In order to see the full picture, we need to consider ν measurements of M.

- We have to look for the average of the measured values

$$\overline{m} = \sum_{n=1}^{\nu} m_k.$$

- Then, if the measured probability distributions fulfill certain conditions, we can estimate the parameter with a precision

$$(\Delta\theta)^2 = \frac{1}{\nu}(\Delta\theta)^2_M = \frac{1}{\nu}\frac{(\Delta M)^2}{|\partial_\theta\langle M\rangle|^2}.$$

[L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, "Quantum metrology with nonclassical states of atomic ensembles," Rev. Mod. Phys. 90, 035005 (2018).]

Magnetometry with the fully polarized state VI

- We consider the fully polarized states of N spin-1/2 particles

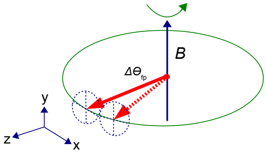
$$|+\tfrac{1}{2}\rangle^{\otimes N}.$$

- For this state,

$$\langle J_z\rangle = \frac{N}{2},\ \langle J_x\rangle = 0,\ (\Delta J_x)^2 = \frac{N}{4},\ \langle J_z\rangle = \frac{N}{2}\cos\theta,\ \langle J_x\rangle = \frac{N}{2}\sin\theta.$$

- We measure the operator

$$M = J_x.$$



- It is not like a classical clock arm, we have a nonzero uncertainty

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta M)^2}{|\partial_\theta\langle M\rangle|^2} = \frac{1}{\nu}\frac{(\Delta J_x)^2}{\langle J_z\rangle^2} = \frac{1}{\nu}\frac{1}{N}.$$

Magnetometry with the fully polarized state VII

- Main result:

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

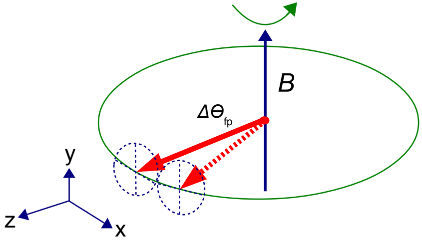
- In some cold gas experiment, we can have $10^3 - 10^{12}$ particles.

- Later we will see that with a separable quantum state we cannot have a better precision.

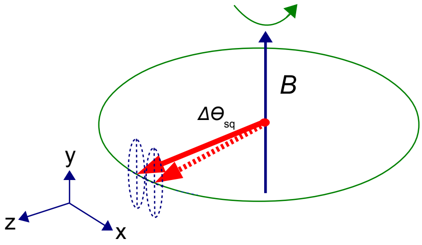
Outline

Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing



fully polarized state (fp)



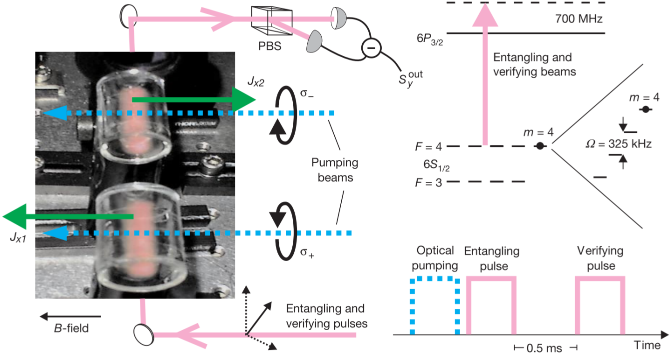
spin-squeezed state (sq)

$\Delta\theta_{\mathrm{fp}}$ and $\Delta\theta_{\mathrm{sq}}$ are the minimal angle difference we can measure.

We can reach

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

Spin squeezing in an ensemble of atoms via interaction with light

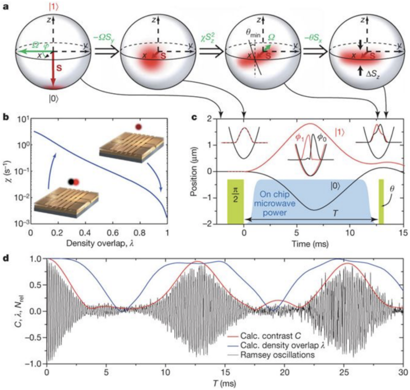


10^{12} atoms, room temperature.

Julsgaard, Kozhekin, Polzik, Nature 2001.

Spin squeezing in a Bose-Einstein Condensate via interaction between the particles

Figure 1: Spin squeezing and entanglement through controlled interactions on an atom chip.

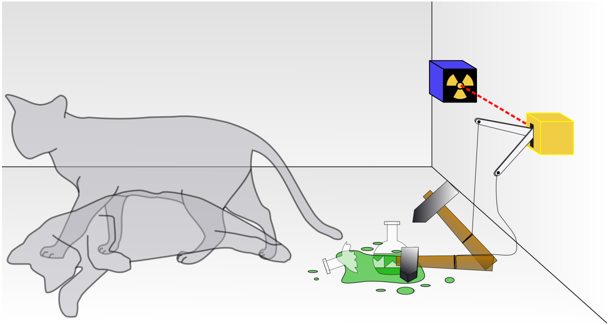


M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature 464, 1170-1173 (2010).

Outline

GHZ state=Schrödinger cat state

- A superposition of two macroscopically distinct states



GHZ state

Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000...00\rangle + |111...11\rangle).$$

- Superposition of all atoms in state "0" and all atoms in state "1".

Metrology with the GHZ state

- Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000...00\rangle + |111...11\rangle),$$

- Unitary

$$|\Psi\rangle(\theta) = U_\theta|\mathrm{GHZ}_N\rangle, \qquad U_\theta = e^{-iJ_z\theta}.$$

- Dynamics

$$|\Psi\rangle(\theta) = \tfrac{1}{\sqrt{2}}(|000...00\rangle + e^{-iN\theta}|111...11\rangle),$$

Metrology with the GHZ state II

- We measure

$$M = \sigma_x^{\otimes N},$$

which is the parity in the x-basis.

- Expectation value and variance

$$\langle M \rangle = \cos(N\theta), \qquad (\Delta M)^2 = \sin^2(N\theta).$$

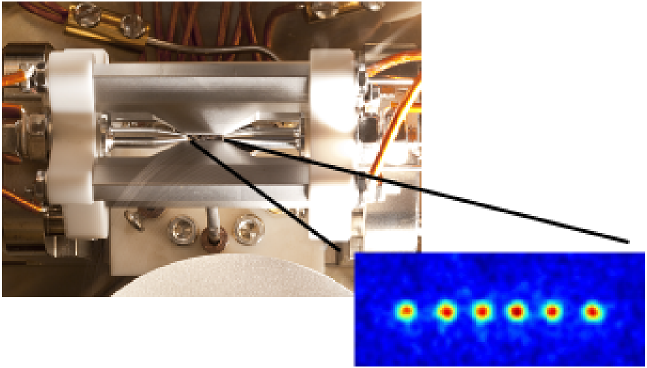
- For $\theta \approx 0$, the precision is

$$(\Delta\theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu N^2}.$$

[e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);

ions: C. Sackett et *al.*, Nature 404, 256 (2000).]

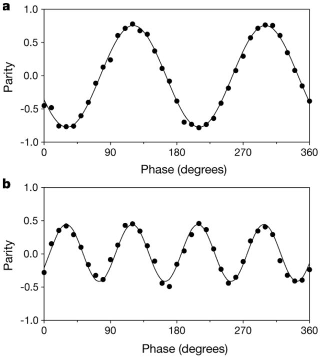
Metrology with the GHZ state III



Quantum Computation with Trapped Ions, Innsbruck

Metrology with the GHZ state IV

Figure 2: Determination of $\rho_{(\uparrow\downarrow)}$.



a, Interference signal for two ions; **b**, four ions. After the entanglement operation of Fig. 1, an analysis pulse with relative phase φ is applied on the single-ion $|\downarrow\rangle \leftrightarrow |\uparrow\rangle$ transition. As φ is varied, the parity of the N ions oscillates as $\cos N\varphi$, and the amplitude of the oscillation is twice the magnitude of the density-matrix element $\rho_{(\uparrow\downarrow)}$. Each data point represents an average of 1,000 experiments, corresponding to a total integration time of roughly 10 s for each graph.

For four ions the curve oscillates faster than for two ions.

[ions: C. Sackett et *al.*, Nature 404, 256 (2000).]

Metrology with the GHZ state IV

- We reached the Heisenberg-limit

$$(\Delta\theta)^2 = \frac{1}{\nu N^2}.$$

- The fully polarized state reached only the shot-noise limit

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Metrology with Dicke states. Clock arm = noise

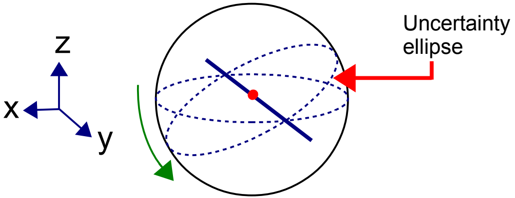
- For our symmetric Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ.
  (We cannot measure first moments, since they are zero.)



Metrology with Dicke states

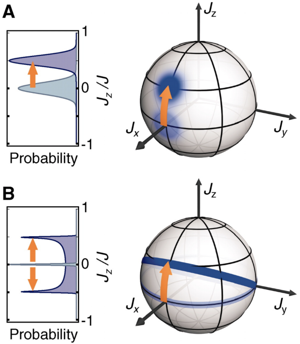
- Dicke states are more robust to noise than GHZ states. (Even if they loose a particle, they remain entangled).
- Dicke states can also reach the Heisenberg-scaling like GHZ states.

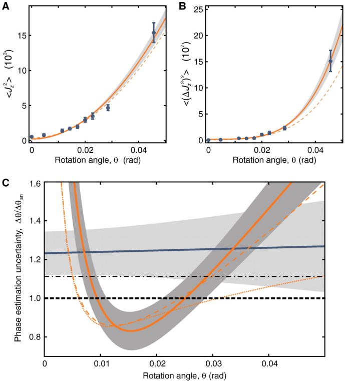
[Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klempt, Science 2011.]

[Metrology with photons: R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.]

Metrology with Dicke states II

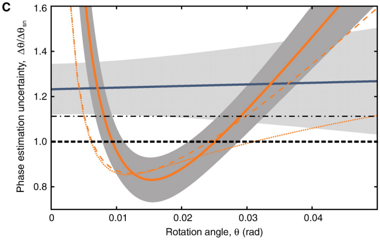
Experiment with cold gas of 8000 atoms.





[Lücke M. Scherer, Kruse, Pezzé, Deuretzbacher, Hyllus, Topic, Peise, Ertmer, Arlt, Santos, Smerzi, Klempt, Science 2011.]

Metrology with Dicke states III

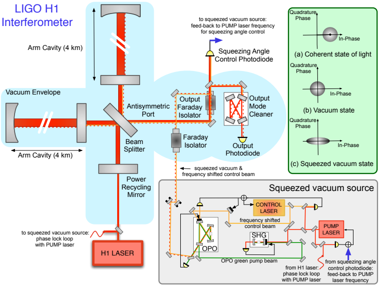


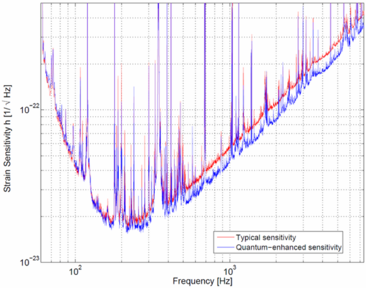
- $\Delta\theta$ is the precision of estimating θ.
- $\Delta\theta_{\mathrm{sn}}$ means the "shot-noise" uncertainty. This is the smallest uncertainty that could be achieved by separable states.
- Black dashed line = the level corresponding to $\Delta\theta/\Delta\theta_{\mathrm{sn}} = 1$.
- Orange solid line = precision of estimating the angle θ in the experiment.
- The orange solid line is below the black dashed line around $\theta \approx 0.015$. Hence, the uncertainty $\Delta\theta$ is smaller than that could be achieved by any separable state, and hence the state of the system is entangled.

Outline

LIGO gravitational wave detector

The performance was enhanced with squeezed light.





The role of clock arm is played by the squeezed coherent state.

[J. Aasi et al., Nature Photonics 2013.]

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

k-producibility/k-entanglement

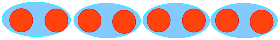
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

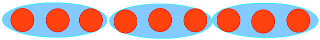
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., Gühne, GT, NJP 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

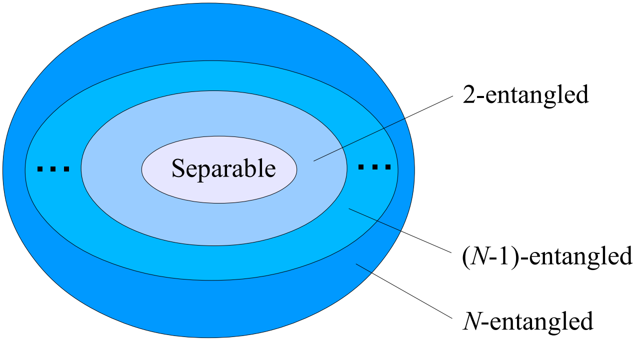


2-entangled



3-entangled

k-producibility/k-entanglement II



$$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \quad \text{2-entangled}$$

$$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \quad \text{3-entangled}$$

$$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle) \quad \text{4-entangled}$$

Outline

1. **Motivation**
   - Why is quantum metrology interesting?
2. **Simple examples of quantum metrology**
   - Classical case: Clock arm
   - Quantum case: Single spin-1/2 particle
   - Magnetometry with the fully polarized state
   - Magnetometry with the spin-squeezed state
   - Metrology with the GHZ state
   - Dicke states
   - Interferometry with squeezed photonic states
3. **Entanglement theory**
   - Multipartite entanglement
   - The spin-squeezing criterion
4. **Quantum metrology using the quantum Fisher information**
   - Quantum Fisher information
   - Quantum Fisher information in linear interferometers
   - Noise and imperfections

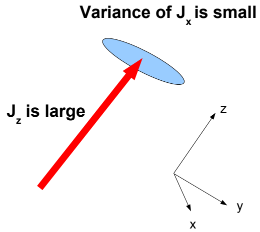
The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

$$\xi_s^2 = N\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2}.$$

If $\xi_s^2 < 1$ then the state is entangled.
[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:



Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k-producible states is

$$(\Delta J_x)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{J_{\max}} \right),$$

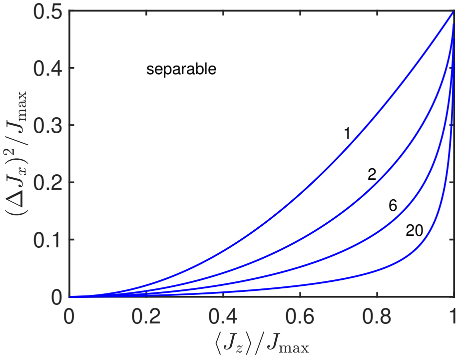
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(Z) := \frac{1}{j} \min_{\frac{\langle j_z \rangle}{j} = Z} (\Delta j_x)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

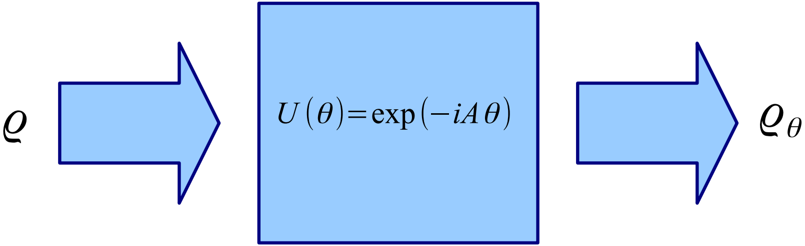
Our experience so far

- We find that more spin squeezing/better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

For the variance of the parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]}$$

holds, where ν is the number of repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The bound includes any estimation strategy, even POVM's.
- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

$$F_Q[|\Psi\rangle, A] = 4(\Delta A)^2{}_\Psi.$$

- For mixed states, it is convex

$$F_Q[\varrho, A] \leq \sum_k p_k F_Q[|\Psi_k\rangle, A],$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Quantum Fisher information - Some basic facts

- The larger the quantum Fisher information, the larger the achievable precision.

- For the totally mixed state it is zero for any A

$$F_Q[\varrho_{\rm cm}, A] = 0,$$

  where $\varrho_{\rm cm} = \mathbb{1}/d$ is the completely mixed state and d is the dimension.

- This is logical: the completely mixed states does not change under any Hamiltonian.

- For any state ϱ that commutes with A, i.e., $\varrho A - A\varrho = 0$ we have

$$F_Q[\varrho, A] = 0.$$

Quantum Fisher information and the fidelity

The quantum Fisher information appears in the Taylor expansion of F_B

$$F_B(\varrho, \varrho_\theta) = 1 - \theta^2 \frac{F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

where

$$\varrho_\theta = \exp(-iA\theta)\varrho\exp(+iA\theta).$$

- Bures fidelity

$$F_B(\varrho_1, \varrho_2) = \mathrm{Tr}\left(\sqrt{\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_1}}\right)^2.$$

- Clearly,

$$0 \le F_B(\varrho_1, \varrho_2) \le 1.$$

  The fidelity is 1 only if $\varrho_1 = \varrho_2$.

Outline

Magnetometry with a linear interferometer

- The Hamiltonian A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

  There are no interaction terms.

- The dynamics rotates all spins in the same way.

Quantum Fisher information for separable states

- Let us consider a pure product state of N qubits

$$|\Psi\rangle_{\rm prod} = |\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes ... \otimes |\Psi^{(N)}\rangle.$$

- Since this is a pure state, we have $F_Q[\varrho, J_l] = 4(\Delta J_l)^2{}_{|\Psi\rangle_{\rm prod}}$.

- Then, for the product state we have

$$(\Delta J_l)^2{}_{|\Psi\rangle_{\rm prod}} = \sum_{n=1}^{N} (\Delta j_l^{(n)})^2{}_{|\Psi^{(n)}\rangle} \leq N \times \frac{1}{4},$$

where we used that for qubits $(\Delta j_l^{(n)})^2 \leq 1/4$.

- Since the quantum Fisher information is convex in the state, the bound is also valid for a mixture of product states, i.e., separable states

$$F_Q[\varrho, J_l] \leq N.$$

The quantum Fisher information vs. entanglement

- For separable states of N spin-1/2 particles (qubits)

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-qubit entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

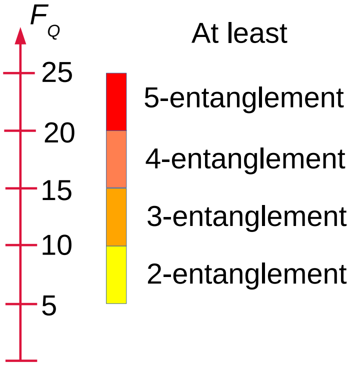
[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states of N qubits

$$F_Q[\varrho, J_l] \leq N^2.$$

The quantum Fisher information vs. entanglement

5 spin-1/2 particles



(Using the $F_Q[\varrho, J_l] \leq kN$. Note that there is a slightly better bound.)

Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N}, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$(\Delta\theta)^2 \geq \frac{1}{\nu k N}.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N^2}.$$

Outline

Noisy metrology: Simple example

- A particle with a state ϱ_1 passes trough a map that turns its internal state to the fully mixed state with some probability p as

$$\epsilon_p(\varrho_1) = (1-p)\varrho_1 + p\frac{\mathbb{1}}{2}.$$

- This map acts in parallel on all the N particles.

- Metrology with a spin squeezed state

$$(\Delta\theta)^2 = \frac{1}{\nu}\frac{(\Delta J_x)^2}{\langle J_z\rangle^2} \geq \frac{1}{\nu}\frac{\frac{pN}{4}}{\frac{N^2}{4}} = p\frac{1}{\nu N} \propto {\color{red}\frac{1}{\nu N}}.$$

- Shot-noise scaling if $p > 0$.

[G. Toth, and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).]

Noisy metrology: General treatment

- In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

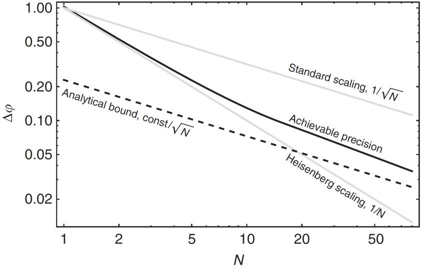


Figure from
[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

- Correlated noise is different.

Take home message

- Quantum physics makes it possible to obtain bounds for precision of the parameter estimation in realistic many-particle quantum systems.

- Shot-noise limit: Non-entangled states lead to $(\Delta\theta)^2 \geq \frac{1}{\nu N}$.

- Heisenberg limit: Fully entangled states can lead to $(\Delta\theta)^2 = \frac{1}{\nu N^2}$.

- At the end, noise plays a central role.

Reviews

- M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. 7, 125 (2009).
- V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics 5, 222 (2011).
- C. Gross, Spin squeezing, entanglement and quantum metrology with Bose-Einstein condensates, J. Phys. B: At.,Mol. Opt. Phys. 45, 103001 (2012).
- R. Demkowicz-Dobrzanski, M. Jarzyna, and J. Kolodynski, Chapter four-quantum limits in optical interferometry, Prog. Opt. 60, 345 (2015).
- L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: fundamentals and applications in quantum metrology, Rev. Mod. Phys. 90, 035005 (2018).

Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,

Quantum metrology from a quantum information science perspective,

J. Phys. A: Math. Theor. 47, 424006 (2014),
special issue "50 years of Bell's theorem"
(open access).