

**General characteristics of multi-partite  
quantum systems**  
**(Lecture of the Quantum Information class of  
the Master in Quantum Science and  
Technology)**

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## 1 General characteristics of multi-partite quantum systems

- A. Classical bits
- B. Quantum bit - pure states
- C. Multi-qubit systems - pure states
- D. Measurement
- E. Mixed states and the density matrix
- F. Geometry of quantum states
  - A single qubit
  - A single qudit (qunit):  $d$ -dimensional systems
- G. Two or more qubits: reduced states
- H. Purifications
- I. Purity
- J. Entropy
  - Shannon entropy
  - Von Neumann entropy
  - Quantum relative entropy
  - Linear entropy
- K. Fidelity

# A single classical bit

- A classical bit can be either 0 or 1. Can we still use it to describe a real number between 0 and 1?
- For that, we need an ensemble of several classical bits

$$\{b_k\}_{k=1}^M, \quad (1)$$

where  $b_k = 0$  or  $1$

- We can interpret the average value and the variance. That is,

$$\langle b \rangle = \frac{1}{M} \sum_k b_k, \quad (2)$$

and

$$(\Delta b)^2 = \frac{1}{M} \sum_k (b_k - \langle b \rangle)^2. \quad (3)$$

## A single classical bit II

- This can also be given with probabilities:
- Let  $P_0$  and  $P_1$  be the probabilities of having a 0 or a 1.
- The expectation value and the variance are the function of  $P_0$  and  $P_1$ . Since  $P_0 + P_1 = 1$ , we have a **single real degree of freedom** that describes the statistical properties of an ensemble of bits.
- Hence,

$$\langle b \rangle = P_1 \quad (4)$$

and

$$(\Delta b)^2 = P_0(0 - P_1)^2 + P_1(1 - P_1)^2. \quad (5)$$

# Stochastic computing

- Stochastic computing uses random bits to calculate (John von Neumann, 1953).
- A random bit represents a real number between 0 and 1. Two random bits can easily be multiplied.

$$\langle b_1 b_2 \rangle = \langle b_1 \rangle \langle b_2 \rangle. \quad (6)$$

- We need many samples to get the average with small error.

Lectures on  
PROBABILISTIC LOGICS AND THE SYNTHESIS OF RELIABLE  
ORGANISMS FROM UNRELIABLE COMPONENTS

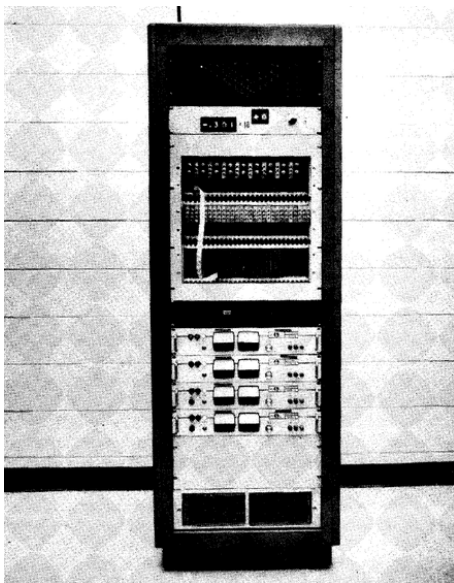
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at the

# Stochastic computing III



The RASCEL stochastic computer, circa 1969, Wikipedia.

# Stochastic computing IV

Multiplication is possible with an AND gate.

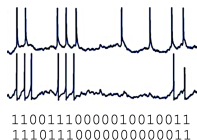


Figure 1.2: Similarity of biological signals and stochastic numbers; information is carried via pulses.

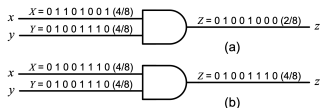


Figure 1.3: Stochastic multiplication: (a) accurate result with uncorrelated inputs; (b) inaccurate result due to correlated inputs.



## Several classical bits

- $N$  classical bits can be in one of the  $2^N$  binary states. For example, for  $N = 2$ , these are 00, 01, 10 and 11.
- For  $N = 2$ , these are

$$P_{00}, P_{01}, P_{10}, P_{11}. \quad (7)$$

- The ensemble of the  $N$ -bit units can be described by the  $2^N$  probabilities.
- Since, again, the sum of all the probabilities is 1, **we need**  $2^N - 1$  **real degrees of freedom** to describe the statistical properties of such an ensemble.

## Several classical bits II

- Let us consider some function of  $N$  bits  $f(k)$ , where  $k$  is now an  $N$  bit number.
- Then, the expectation value of  $f$  is

$$\langle f \rangle = \sum_{k=0}^{2^N-1} p_k f(k) = \vec{p} \vec{f}, \quad (8)$$

where  $k$  is an  $N$ -bit number, i.e., an integer between 0 and  $2^N - 1$ . We put the  $f_k$ 's into a vector  $\vec{f}$ . We also put the  $p_k$  probabilities into  $\vec{p}$ .

## Several classical bits III

- We can also write

$$\langle f^2 \rangle = \sum_k p_k f_k^2 \quad (9)$$

Hence,

$$(\Delta f)^2 = \sum_k p_k f_k^2 - \left( \sum_k p_k f_k \right)^2. \quad (10)$$

These were relevant, since in the quantum case, we will have similar expressions.

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# Quantum bit - pure states

- A quantum bit (=two-state system, spin- $\frac{1}{2}$  particle) can be in a pure state

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (11)$$

where  $\alpha$  and  $\beta$  are complex numbers, and the normalisation condition  $|\alpha|^2 + |\beta|^2 = 1$ .

- Note that the overall phase does not matter, thus a pure quantum bit is described by two degrees of freedom.
- The two complex coefficients have **4 real degrees of freedom**.
- However, due to the normalisation condition and the arbitrariness of the overall phase we are left with **two degrees of freedom**.)

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## Multi-qubit systems - pure states

- What about a two-qubit system? What kind of states it can be in?  
One could think on qubit 1 in state

$$|q_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle, \quad (12)$$

and qubit 2 in state

$$|q_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle. \quad (13)$$

- However, we all know that the general state of the two-qubit system can be given as

$$|q_{12}\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle. \quad (14)$$

## Multi-qubit systems - pure states II

- In general, for  $N$  qubits we need  $N$  complex numbers. Again the state has to be normalized and the overall phase does not matter, thus this means  $2 \times 2^N - 2$  real degrees of freedom.
- We can place the coefficients in a vector, called state vector and write

$$|\Psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}. \quad (15)$$

- The properties of the state vector are: it is normalized

$$\langle\Psi|\Psi\rangle = 1. \quad (16)$$



## Multi-qubit systems - pure states III

- An overall phase does not matter:

$$e^{-i\theta}|\Psi\rangle \quad (17)$$

describes the same state for any  $\theta$ .

- The expectation value of an operator for a pure state can be obtained as

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle = \text{Tr}(A |\Psi\rangle \langle \Psi|). \quad (18)$$

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# Measurement

- The von Neumann measurement in the  $z$  basis results is either 0 or 1. If the state was  $\alpha|0\rangle + \beta|1\rangle$ , then we get a statistical mixture of 0 and 1, with the probabilities

$$P_0 = |\alpha|^2, \quad (19)$$

and

$$P_1 = |\beta|^2. \quad (20)$$

That is, from an ensemble of quantum bits we get an ensemble of classical bits.

- If we measure in the  $x$  basis, we get another classical ensemble.
- For a multi-qubit system, if we measure in the some basis (e.g.,  $x$ ,  $y$  or  $z$ ), we get an ensemble of  $N$ -bit systems. However, for each choice of basis we get a different classical ensemble.

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# Mixed states and the density matrix

- So far we were talking about pure states.
- In reality, in an experiment we do not have a situation where a machine always produces the  $|\Psi_1\rangle$  state.
- Sometimes it makes mistakes, and produces the  $|\Psi_k\rangle$  states for  $k = 2, 3, \dots$  How to describe such a situation?

$ \Psi_1\rangle$	$p_1$
$ \Psi_2\rangle$	$p_2$
$ \Psi_3\rangle$	$p_3$
...	...

# Mixed states and the density matrix

- What is the expectation value of an operator in such a system?  
We can write it as

$$\langle A \rangle = \sum_k p_k \langle \psi_k | A | \psi_k \rangle = \text{Tr} \left( A \sum_k p_k | \psi_k \rangle \langle \psi_k | \right). \quad (21)$$

- This can be rewritten as

$$\langle A \rangle = \text{Tr}(\rho A), \quad (22)$$

where

$$\rho = \sum_k p_k | \psi_k \rangle \langle \psi_k | \quad (23)$$

is the density matrix (Neumann, Landau).

- Note that if  $\rho$  is diagonal, we obtain

$$\langle A \rangle = \text{Tr}(\rho A) = \sum_k \rho_{kk} A_{kk}. \quad (24)$$

That is,  $A$  is written in the eigenbasis of  $\rho$ . This is the scalar product of two vectors as in  $\langle f \rangle = \vec{p} \vec{f}$  [given in Eq. (8)].

## Mixed states and the density matrix II

- The density matrix describes the state completely. Now we see, why the overall phase does not matter:

$$e^{-i\theta}|\Psi_k\rangle\langle\Psi_k|e^{+i\theta} = |\Psi_k\rangle\langle\Psi_k|. \quad (25)$$

- The properties of the density matrix are

$$\begin{aligned} \rho &= \rho^\dagger, \\ \rho &\geq 0, \\ \text{Tr}(\rho) &= 1. \end{aligned} \quad (26)$$

- **A  $2^N \times 2^N$  density matrix has  $4^N - 1$  real parameters.**
- For  $N = 1$ , this means 3 real parameters, corresponding to the three coordinates of the Bloch vector. For  $N = 2$ , this means 8 real parameters.

## Mixed states and the density matrix III

- We can also say that

$$\text{Tr}(\rho^2) \leq 1. \quad (27)$$

It is one only for pure (rank-1) states.

- The density matrix can be decomposed into the sum of pure states in many ways. The decomposition

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \quad (28)$$

is not unique, i.e., it is not necessarily an eigendecomposition. This has a large importance for entanglement theory.



# Mixed states and the density matrix IV

Summary:

	$N$ bits	$N$ qubits
Number of DOF	$2^N - 1$	$4^N - 1$
Description	$\vec{p}$	$\rho$
Expectation value	$\vec{f}\vec{p}$	$\text{Tr}(A\rho)$
Normalization	$\sum_k p_k = 1$	$\text{Tr}(\rho) = 1$

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# Bloch vector

- For a single qubit, **the density matrix has three real parameters**. It can be written as

$$\varrho = \frac{1}{2} \left( \mathbb{1} + \sum_{l=x,y,z} v_l \sigma_l \right), \quad (29)$$

where  $\sigma_l$  are the Pauli spin matrices.

- Using  $\text{Tr}(\sigma_k \sigma_l) = 2\delta_{kl}$ , we can write

$$\text{Tr}(\varrho^2) = \frac{1}{2} + \frac{1}{2} \sum_{l=x,y,z} v_l^2. \quad (30)$$

That is, the Bloch vector has a maximal length for pure states.

## Bloch vector II

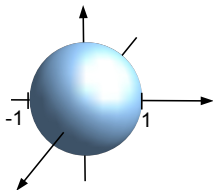
- From  $\text{Tr}(\rho^2) \leq 1$ , the condition for being physical is Eq. (26), which is equivalent to

$$\sum_{l=x,y,z} |v_l|^2 \leq 1. \quad (31)$$

The three-element vector is called the Bloch vector.

## Bloch vector III

- Let us identify the points in  $(v_x, v_y, v_z)$  corresponding to physical states. They are in a ball.
- The pure states are on the surface.
- Mixed states are inside the Ball. This is because  $\text{Tr}(\rho^2)$  is directly related to the length of the Bloch vector.
- The  $|0\rangle$  and  $|1\rangle$  correspond to the North and South Pole.
- $|0\rangle + \exp(-i\phi)|1\rangle$  correspond to points on the equator.



Set of physical quantum states for a single qubit. The axes correspond to  $v_l$  for  $l = x, y, z$ . Pure states correspond to points on the surface, mixed states correspond to internal points.

## A single qudit (qunit): $d$ -dimensional systems

- For higher dimensional systems the picture is much more complicated. Let us consider qudits with dimension  $d$ .
- Similarly to the case before, **a  $d \times d$  Hermitian matrix with a unit trace has  $d^2 - 1$  degrees of freedom.**
- Hence, we can write a density matrix as a linear combination of  $d^2 - 1$  SU( $d$ ) generators as

$$\rho = \frac{1}{d} \mathbb{1} + \frac{1}{2} \sum_{l=1}^{d^2-1} v_l g_l. \quad (32)$$

Here,

$$\text{Tr}(g_k g_l) = 2\delta_{kl}. \quad (33)$$

(Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices.  $d = 3$  : Gell-Mann matrices.)

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- Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices.  $d = 3$  : for instance, Gell-Mann matrices.

# A single qudit (qunit): $d$ -dimensional systems II

- Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

There are other possibilities: J. Lawrence, quant-ph/0403095.

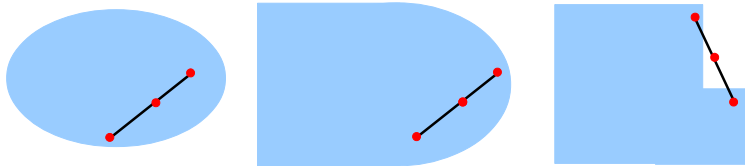


## A single qudit (qunit): $d$ -dimensional systems III

- Let us again look at the points  $(v_1, v_2, \dots, v_{d^2-1})$  corresponding to physical states.
- First note that the set is convex. This is because mixing two physical states  $\varrho_1$  and  $\varrho_2$ , we always get a physical state

$$\varrho = p\varrho_1 + (1 - p)\varrho_2. \quad (36)$$

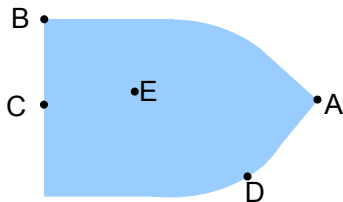
# A single qudit (qunit): $d$ -dimensional systems IV



Two convex objects and one that is not convex.

# A single qudit (qunit): $d$ -dimensional systems V

- On the next figure we will show the set of quantum states.
- The coordinate axis could be the  $v_l$ , for example.
- Inside the set there are the density matrices with full rank.
- On the boundary there are the states with less than full rank, such as for example rank-1 states, which are pure states.



Set of physical quantum states. Note that the set is convex.  
A,B,D: rank-1 states. C: rank-2 state. E: full rank states.

# A single qudit (qunit): $d$ -dimensional systems VI

- **Observation.** The following inequality is true

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B). \quad (37)$$

*Proof.* Let us consider that for a Hermitian matrix  $X$  we have

$$\lambda_{\min}(X) = \min_{\psi} \langle \psi | X | \psi \rangle. \quad (38)$$

Then, for  $A$  and  $B$  Hermitian matrices we have

$$\begin{aligned} \lambda_{\min}(A + B) &= \min_{\psi} \langle \psi | A + B | \psi \rangle \geq \min_{\psi} \langle \psi | A | \psi \rangle + \min_{\psi} \langle \psi | B | \psi \rangle \\ &= \lambda_{\min}(A) + \lambda_{\min}(B). \end{aligned} \quad (39)$$

□

We can prove similarly that

$$\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B). \quad (40)$$

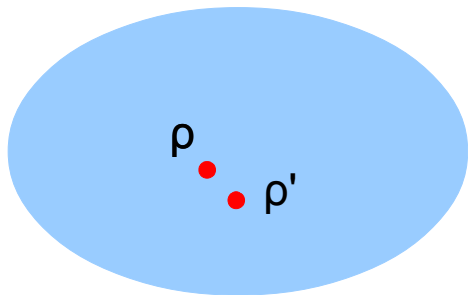
# Full rank states

- Using this, we can say the following.
- **Observation.** Full-rank states are inside the set.  
*Proof.* If the state is full rank, it means that for some small  $\epsilon$

$$\varrho' = \varrho + \epsilon H \tag{41}$$

is also physical, where  $H$  is a trace 0 Hermitian matrix. Why is that? See also the next figure.

## Full rank states II



We take an internal state  $\rho$  and consider the states  $\rho'$  in its neighborhood.

# Full rank states III

- It is physical since
  - 1 Trace is 1.
  - 2 Hermitian.
  - 3 Eigenvalues are nonzero for small epsilon. This is because

$$\lambda_{\max}(\rho) + \lambda_{\max}(\epsilon H) \geq \lambda_k(\rho') \geq \lambda_{\min}(\rho) + \lambda_{\min}(\epsilon H). \quad (42)$$

Here we have

$$\lambda_{\min}(\epsilon H) = \begin{cases} +\epsilon \lambda_{\min}(H), & \text{if } \epsilon \geq 0, \\ -|\epsilon| \lambda_{\max}(H), & \text{if } \epsilon < 0. \end{cases} \quad (43)$$

Similar statement holds for  $\lambda_{\max}(\epsilon H)$ .  $\square$

# Non-full-rank states

- **Observation.** Non-full-rank states are on the surface of the set.
- *Proof.* If the state is not full rank, then it has zero eigenvalues. Thus, there is an  $H$  such that  $\varrho'$  is aphysical for any  $\epsilon > 0$  or any  $\epsilon < 0$ .
- To be more explicit, let us write

$$\varrho = UDU^\dagger, \quad (44)$$

such that  $D$  contains the eigenvalues. Here,

$$D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_d), \quad (45)$$

and the eigenvectors are

$$U = [|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, \dots, |\Psi_d\rangle]. \quad (46)$$



## Non-full-rank states II

- Assume that  $\lambda_d = 0$ . Then,

$$\varrho' = \varrho + \epsilon(|\Psi_d\rangle\langle\Psi_d| - \mathbb{1}/d) \quad (47)$$

has a negative eigenvalue for any  $\epsilon < 0$ . The Identity is needed to make the expression zero-trace.

- This is because the eigenvalues of this matrix are given by

$$D' = \text{diag}(\lambda_1 - \epsilon/d, \lambda_2 - \epsilon/d, \lambda_3 - \epsilon/d, \dots, \lambda_d + \epsilon(1 - 1/d)). \quad (48)$$

□

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## Two or more qubits: reduced states

- How can one see the state of a qubit, if it is the part of an entangled state?
- A reduced state of a bipartite system can be obtained after tracing out one of the subsystems. Let us consider a two-qubit system and write the density matrix in the basis  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ . Then, denote the elements of the density matrix by

$$\rho_{ij,kl}, \quad (49)$$

where  $i, j, k, l = 0, 1$ . In other words, it looks like

$$\rho = \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{pmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{pmatrix}. \quad (50)$$

Thus, the size of the density matrix is  $4 \times 4$ .

## Two or more qubits: reduced states II

- To become familiar with bras and kets, one can even use the completeness relation

$$\text{Identity} = \sum_{ij} |ij\rangle\langle ij|. \quad (51)$$

Then, one obtains

$$\text{Identity} \times \rho \times \text{Identity} = \sum_{ijkl} |ij\rangle\langle ij|\rho|kl\rangle\langle kl|, \quad (52)$$

where the expression in the bracket is just the matrix element of the density matrix

$$\rho_{ij,kl} = \langle ij|\rho|kl\rangle. \quad (53)$$

Hence, the density matrix can be written as

$$\rho = \sum_{ijkl} \rho_{ij,kl} |ij\rangle\langle kl|. \quad (54)$$

## Two or more qubits: reduced states III

- Then, tracing out the second subsystem gives the reduced state

$$\text{Tr}_2(\rho) = \rho_{\text{red}}, \quad (55)$$

which is given as

$$\rho_{\text{red},ik} = \sum_m \rho_{im,km}. \quad (56)$$

This is a 2x2 density matrix of a qubit. With this

$$\langle \mathbf{A} \otimes \mathbb{1} \rangle_{\rho} = \langle \mathbf{A} \rangle_{\rho_{\text{red}}}. \quad (57)$$

- Graphical representation: in the blockdiagonal representation, we sum the elements in the diagonal of the small matrices.
- Tracing out for pure states:

$$\text{Tr}_2 \left( \sum_k \alpha_k |\psi_k\rangle \langle \phi_k| \right) = \sum_k |\alpha_k|^2 |\psi_k\rangle \langle \psi_k|. \quad (58)$$

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# Purifications

- The pure state  $|\Psi_{AB}\rangle$  is the purification of the mixed state  $\rho_A$  if

$$\text{Tr}_B(|\Psi_{AB}\rangle\langle\Psi_{AB}|) = \rho_A. \quad (59)$$

Note that  $|\Psi_{AB}\rangle$  lives on subsystems  $A$  and  $B$ , while  $\rho_A$  lives on subsystem  $A$  only.

- Let us assume that a density matrix is defined as

$$\rho_A = \sum_k p_k |\phi_k\rangle\langle\phi_k|_A. \quad (60)$$

- Then, a purification can be a pure state

$$|\Psi\rangle_{AB} = \sum_k \sqrt{p_k} |\phi_k\rangle_A \otimes |k\rangle_B, \quad (61)$$

where  $|k\rangle_B$  denotes an orthonormal basis of the subsystem  $B$ .

# Purifications II

- If  $|\Psi\rangle_{AB}$  is a purification then

$$|\Psi\rangle'_{AB} = \mathbb{1}_A \otimes U_B |\Psi\rangle_{AB}, \quad (62)$$

is also a purification.



## 1 General characteristics of multi-partite quantum systems

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- B. Quantum bit - pure states
- C. Multi-qubit systems - pure states
- D. Measurement
- E. Mixed states and the density matrix
- F. Geometry of quantum states
  - A single qubit
  - A single qudit (qunit):  $d$ -dimensional systems
- G. Two or more qubits: reduced states
- H. Purifications
- **I. Purity**
- J. Entropy
  - Shannon entropy
  - Von Neumann entropy
  - Quantum relative entropy
  - Linear entropy
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# Purity

- Defined as

$$\text{Tr}(\rho^2). \quad (63)$$

- 1 for pure states.
- $1/d$  for the completely mixed state.

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# Shannon entropy

- There is a source that outputs an integer number between 1 and  $d$ .
- The Shannon entropy is given as

$$H = - \sum_{k=1}^d p_k \log p_k. \quad (64)$$

# Shannon entropy II

- Properties
  - Classical, not quantum.
  - In information theory, the entropy of a random variable is the average level of "information", "surprise", or "uncertainty" inherent in the variable's possible outcomes (wikipedia). That is, the source can have  $d$  possible outputs with some probability. Relation to compression of data.
  - $H = 0$  if  $p_1 = 1$ , all other  $p_k = 0$ .  $\vec{p} = (1, 0, 0, 0, \dots)$ . The output is always the same. No information is provided. Comment: we can show that, using L'Hospital's rule,

$$\lim_{x \rightarrow 0} (x \log x) = \lim_{x \rightarrow 0} \frac{\log x}{1/x} = \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = - \lim_{x \rightarrow 0} x = 0. \quad (65)$$

- $H = \log d$  (maximal) if  $p_k = \frac{1}{d}$ .  $\vec{p} = (\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \dots)$ . All outputs are equally probable, a lot of information is provided.

# Von Neumann entropy

- Von Neumann entropy for a quantum state is defined as

$$S(\rho) = -\text{Tr}(\rho \log \rho) \equiv -\langle \log \rho \rangle. \quad (66)$$

- Note: matrix logarithm! It can be written with the eigenvalues of the density matrix as

$$S(\rho) = -\sum_{k=1}^d \lambda_k \log_2 \lambda_k. \quad (67)$$

# Von Neumann entropy II

- Properties

- Quantum. "Quantum version" of the Shannon entropy.
- For a pure state we have  $\lambda_k = \{1, 0, 0, \dots, 0\}$ , and thus it is zero.
- Its maximal is for the completely mixed state for which  $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}$ , and its value is  $\log_2 d$ .
- Concave, i.e.,  $S(p\rho_1 + (1-p)\rho_2) \geq pS(\rho_1) + (1-p)S(\rho_2)$ . Put the figure of a concave function and explain concavity.
- Additive for independent systems,  $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ .
- Strongly subadditive,  $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$ . The matrices  $\rho_B, \rho_{AB}$ , etc. reduced states.
- Subadditive,  $S(\rho_{AC}) \leq S(\rho_A) + S(\rho_C)$ .
- Often used in condensed matter physics and field theory. See block entropy depending on the block size.

# Quantum relative entropy

- The relative entropy is given as

$$S(\varrho\|\sigma) = -\text{Tr}[\varrho(\log \sigma - \log \varrho)] = -\text{Tr}(\varrho \log \sigma) - S. \quad (68)$$

- Properties

- $S(\varrho\|\sigma) \geq 0$ .
- $S(\varrho\|\sigma) = 0$  if and only if  $\varrho = \sigma$ .
- Not symmetric  $S(\varrho\|\sigma) \neq S(\sigma\|\varrho)$ .
- Sort of a distance between two quantum states.



# Linear entropy

- The linear entropy is defined as

$$S_{\text{lin}}(\rho) = 1 - \text{Tr}(\rho^2) \equiv \langle \mathbb{1} - \rho \rangle. \quad (69)$$

- It is often easier to obtain than the von Neumann entropy.
- Its relation to von Neumann entropy via the Mercator series is

$$-\langle \log \rho \rangle = \langle \mathbb{1} - \rho \rangle + \langle (\mathbb{1} - \rho)^2 \rangle / 2 + \langle (\mathbb{1} - \rho)^3 \rangle / 3 + \dots \quad (70)$$

This is based on expanding

$$\log(\mathbb{1} - (\mathbb{1} - \rho)) \quad (71)$$

Mercator series

$$\log(1 + x) = x - x^2/2 + x^3/3 - + \dots \quad (72)$$

Note that

$$\mathbb{1} - \rho \geq 0. \quad (73)$$

Hence,

$$S \geq S_{\text{lin}}. \quad (74)$$

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# Fidelity

How to measure the distance between quantum states?

- Pure states: overlap square,  $\langle \psi | \phi \rangle^2$ .
- $\langle \psi | \phi \rangle = 0$  if and only if  $|\psi\rangle = |\phi\rangle$ .
- A pure state and a mixed state:

$$\text{Tr}(|\Psi\rangle\langle\Psi|\varrho) = \langle\Psi|\varrho|\Psi\rangle. \quad (75)$$

- Two mixed states: more difficult

$$F(\varrho, \sigma) = \left( \text{Tr}(\sqrt{\sqrt{\varrho}\sigma\sqrt{\varrho}}) \right)^2. \quad (76)$$

- $0 \leq F(\varrho, \sigma) \leq 1$ .
- $F(\varrho, \sigma) = 1$  if and only if  $\varrho = \sigma$ .
- $F(\varrho, \sigma) = 0$  if  $\varrho$  and  $\sigma$  live on orthogonal subspaces.
- Symmetric  $F(\varrho, \sigma) = F(\sigma, \varrho)$ .
- Let us check consistency. If  $\rho = |\Psi\rangle\langle\Psi|$  then  $\sqrt{\varrho} = \varrho = |\Psi\rangle\langle\Psi|$ . Then,

$$F(\varrho, \sigma) = \text{Tr}(\sqrt{|\Psi\rangle\langle\Psi|\sigma|\Psi\rangle\langle\Psi|})^2 = \langle\Psi|\sigma|\Psi\rangle \text{Tr}(\sqrt{|\Psi\rangle\langle\Psi|})^2 = \langle\Psi|\sigma|\Psi\rangle. \quad (77)$$

Hence, we got back the formula for the simpler case.

- Defining the Fidelity with a maximum over purifications

$$F(\varrho, \sigma) = \max_{|\Psi_\sigma\rangle} |\langle \Psi_\varrho | \Psi_\sigma \rangle|^2. \quad (78)$$

- $|\Psi_\varrho\rangle$  is a purification of  $\varrho$ ,  $|\Psi_\sigma\rangle$  is a purification of  $\sigma$ ,