Entanglement theory (entangled/not entangled) (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

1. **Entanglement theory (entangled/not entangled)**
   - Motivation
   - A. Bipartite case
     - Pure states
     - Mixed states
   - B. Entanglement criteria
     - Partial transposition
     - Entanglement witnesses
     - Variance based criteria
   - C. Multipartite case

Entanglement detection

- We would like to distinguish entangled states from separable states.
- The problem is very difficult, there are no general methods.

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Separability and entanglement of pure states

- if the pure state is a product state then it is separable. If it is not a product state then it is entangled.

- If the reduced state
$$\varrho_1 = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)$$
is pure then the state is a product state, otherwise it is entangled. In other words, if
$$\mathrm{Tr}\{[\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)]^2\} = 1$$
then the state is a product state.

Separability and entanglement for pure states I

- A quantum state is called separable if it can be written as a convex sum of product states as

$$\varrho = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_k^{(n)}$ are single-qudit density matrices. A state that is not separable is called entangled.

- R. F. Werner, 1989:

with the density matrix $W = \sum_{r=1}^{n} p_r W_r^1 \otimes W_r^2$, i.e., W is a convex combination of product states. Expectation

Separability and entanglement of mixed states II

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Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model

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A state of a composite quantum system is called classically correlated if it can be approximated by convex combinations of product states, and Einstein-Podolsky-Rosen correlated otherwise. Any classically correlated state can be modeled by a hidden-variable theory and hence satisfies all generalized Bell's inequalities. It is shown by an explicit example that the converse of this statement is false.

I. INTRODUCTION

Consider a composite quantum system described in a Hilbert space $\mathcal{H}=\mathcal{H}^1\otimes\mathcal{H}^2$. An uncorrelated state of this system is given by a density matrix W [i.e., an operator $W\in\mathcal{B}(\mathcal{H})$ with $W\geq 0$ and tr $W=1$] in $\mathcal{H}$ of the form $W=W^1\otimes W^2$ for two density matrices $W^i\in\mathcal{B}(\mathcal{H}_i)$. This is equivalent to saying that the expectation value $\mathrm{tr}(WA_1\otimes A_2)$ for the joint measurement of observables $A^i\in\mathcal{B}(\mathcal{H}^i)$ $(i=1,2)$ on the respective subsystems always factorizes, i.e.,

$$\mathrm{tr}(WA^1\otimes A^2)=\mathrm{tr}(W\cdot A^1\otimes\mathbb{1})\mathrm{tr}(W\cdot\mathbb{1}\otimes A^2)$$
$$=\mathrm{tr}(W^1A^1)\mathrm{tr}(W^2A^2)\ .$$

Such uncorrelated states can be prepared very easily by using two preparing devices for systems 1 and 2, which

it can be approximated (e.g., in trace norm) by density matrices of the form (1). States that are not classically correlated have been called *EPR correlated*[1] to emphasize the crucial role of such states in the Einstein-Podolsky-Rosen paradox, and for the violations of Bell's inequalities (see below). EPR correlation and classical correlation are defined as a property of the density matrix W. Since there are usually very different ways of preparing the same state W, classical correlation does not mean that the state has actually been prepared in the manner described, but only that its statistical properties can be reproduced by a classical mechanism.

The terminology "classically correlated" is further justified by the observation that in classical probability theory all states have this property. States in probability theory are given by probability measures, and the state of a composite system is given by a probability measure on a

Separability and entanglement of pure states III

Comments:

- For pure states it is easy to decide whether a state is separable of not. For mixed states, it is very hard.
- Hand waving meaning of the definition above: with probability p_k a machine produced the product state $\varrho_k^{(1)} \otimes \varrho_k^{(2)}$.
- The two parties (i.e., 1 and 2) can be far from each other (i.e., on the Moon and on Earth).
- No real quantum dynamics is needed between the two parties to create the separable state.

Separability and entanglement of pure states IV

Comments (continued)

- Separable states can be correlated. For example, the state

$$\frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

  has nonzero correlations, however, it is separable.

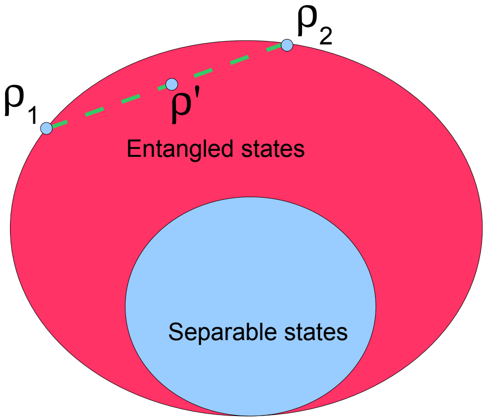
- This can be seen noting that

$$\langle \sigma_z \otimes \sigma_z \rangle = +1.$$

  We can also say that

$$\langle \sigma_z \otimes \sigma_z \rangle - \langle \sigma_z \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes \sigma_z \rangle = +1.$$

Separability and entanglement of pure states V



The set of entangled states and the set of separable states. Again, the set of all states is convex, similarly, as the set of separable states is convex. $\varrho' = p\varrho_1 + (1-p)\varrho_2$.

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     - Variance based criteria
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Entanglement criteria

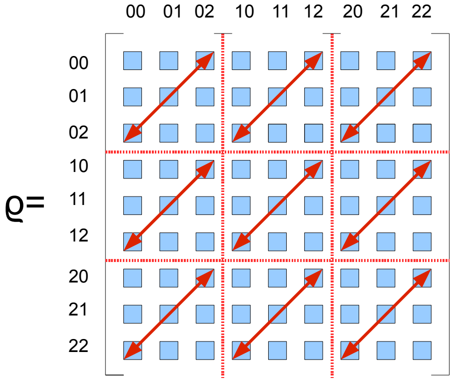
- Deciding whether a state is entangled or not is a difficult problem. There are no necessary and sufficient conditions for entanglement in general.
- However, there are conditions that are necessary and sufficient for small systems.
- There are also conditions that are sufficient conditions for entanglement for larger systems, but does not detect all entangled states.

Partial transposition

- Partial transposition

$$(\varrho^{T1})_{ij,kl} = \varrho_{kj,il}.$$

- Let us see how to do the partial transposition on a system of two qutrits.



Partial transposition II

- Let us take a bipartite separable state

$$\varrho_{\rm sep} = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)}.$$

- Let us carry out the so called partial transposition operation on the second subsystem. Then we get

$$\varrho_{\rm sep}^{\rm T2} = \sum_k p_k \varrho_k^{(1)} \otimes (\varrho_k^{(2)})^T \geq 0.$$

- That is, if all $\varrho_k^{(n)} \geq 0$, then the matrices obtained from them by tensor product and transposition are also positive semidefinite.

Partial transposition III

- However, in general, there are states for which

$$\varrho^{\mathrm{T2}} \ngeq 0.$$

Such states cannot be separable thus they are entangled.

Partial transposition IV

- How to check whether a state is entangled with the Peres-Horodecki criterion?

  1. Take the density matrix.
  2. Calculate the partial transpose.
  3. Calculate its eigenvalues.
  4. If there is a negative eigenvalue, the state is entangled. If not, then we do not know.

- The Peres-Horodecki criterion is necessary and sufficient for 2×2 (qubit-qubit) and 2×3 (qubit-qutrit) systems.
- For larger systems, there are quantum states that are entangled, but not detected by the Peres-Horodecki criterion.

Partial transposition V

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Separability Criterion for Density Matrices

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A quantum system consisting of two subsystems is *separable* if its density matrix can be written as $\rho = \sum_A w_A \rho'_A \otimes \rho''_A$, where ρ'_A and ρ''_A are density matrices for the two subsystems, and the positive weights w_A satisfy $\sum w_A = 1$. In this Letter, it is proved that a necessary condition for separability is that a matrix, obtained by partial transposition of ρ, has only non-negative eigenvalues. Some examples show that this criterion is more sensitive than Bell's inequality for detecting quantum inseparability. [S0031-9007(96)00911-8]

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A striking quantum phenomenon is the inseparability of composite quantum systems. Its most famous example is the violation of Bell's inequality, which may be detected if two distant observers, who independently *measure* subsystems of a composite quantum system, *report* their results to a common site where that information is analyzed [1]. However, even if Bell's inequality is satisfied by a given composite quantum system, there

$$\rho_{m\mu,n\nu} = \sum_A w_A (\rho'_A)_{mn} (\rho''_A)_{\mu\nu} . \qquad (2)$$

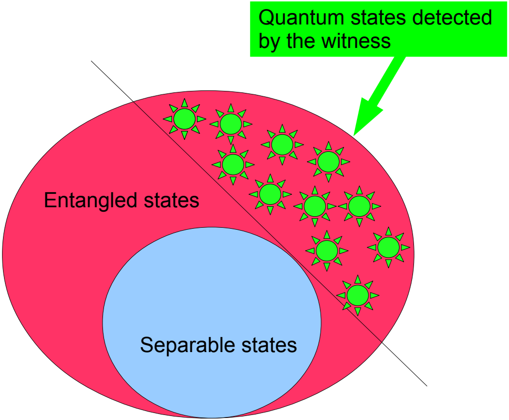
Latin indices refer to the first subsystem, Greek indices to the second one (the subsystems may have different dimensions). Note that this equation can always be satisfied if we replace the quantum density matrices by classical Liouville functions (and the discrete indices are

Entanglement witnesses

Definition. An entanglement witness W is an operator such that

- Its expectation value is nonnegative on all separable states.
- For some entangled state it is negative.

Entanglement witnesses II



Entanglement witnesses. The are entanglement conditons that are linear in operator expectation values.

Example 1

- **Example 1**: Let is see the following entanglement witness

$$W_{xz} = \mathbb{1} - \sigma_x \otimes \sigma_x - \sigma_z \otimes \sigma_z.$$

- Why is this a witness? For product states of the form $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, we have

$$\langle \sigma_x \otimes \sigma_x \rangle + \langle \sigma_z \otimes \sigma_z \rangle = \langle \sigma_x \rangle_{\Psi_1} \langle \sigma_x \rangle_{\Psi_2} + \langle \sigma_z \rangle_{\Psi_1} \langle \sigma_z \rangle_{\Psi_2} \leq 1.$$

- Here, we have to use the Cauchy-Schwarz inequality, that is

$$\vec{v}_1 \cdot \vec{v}_2 \leq |\vec{v}_1||\vec{v}_2|.$$

- Using this we obtain

$$\begin{pmatrix} \langle \sigma_x \rangle_{\Psi_1} \\ \langle \sigma_z \rangle_{\Psi_1} \end{pmatrix} \cdot \begin{pmatrix} \langle \sigma_x \rangle_{\Psi_2} \\ \langle \sigma_z \rangle_{\Psi_2} \end{pmatrix} \leq \sqrt{\langle \sigma_x \rangle_{\Psi_1}^2 + \langle \sigma_z \rangle_{\Psi_1}^2} \sqrt{\langle \sigma_x \rangle_{\Psi_2}^2 + \langle \sigma_z \rangle_{\Psi_2}^2} \leq 1,$$

since the length of Bloch vector is at most 1.

Example 1, II

- Due to the convexity of the set of quantum states, this is also true for separable states. That is

$$\begin{aligned}\langle W\rangle_{\varrho_{\rm sep}} &= {\rm Tr}(W\varrho_{\rm sep}) = {\rm Tr}(W\sum_k p_k\varrho_k^{(1)}\otimes\varrho_k^{(2)}) \\ &= \sum_k p_k {\rm Tr}(W\varrho_k^{(1)}\otimes\varrho_k^{(2)}) \geq 0.\end{aligned}$$

- On the other hand, the maximum for quantum states is 2. Such a maximum is obtained for the state $(|00\rangle + |11\rangle)/\sqrt{2}$.
- How to see this? We need

$$\sigma_x\otimes\sigma_x|00\rangle = |11\rangle,$$
$$\sigma_x\otimes\sigma_x|11\rangle = |00\rangle.$$

Then,

$$\sigma_x\otimes\sigma_x\frac{|00\rangle+|00\rangle}{2} = \frac{|00\rangle+|11\rangle}{2},$$

Hence

$$\langle\sigma_x\otimes\sigma_x\rangle = 1.$$

Example 1, III

We also need

$$\sigma_z \otimes \sigma_z |00\rangle = |00\rangle,$$
$$\sigma_z \otimes \sigma_z |11\rangle = |11\rangle.$$

Then,

$$\sigma_z \otimes \sigma_z \frac{|00\rangle + |11\rangle}{2} = \frac{|00\rangle + |11\rangle}{2}.$$

Hence

$$\langle \sigma_z \otimes \sigma_z \rangle = 1.$$

In summary,

$$\langle \sigma_x \otimes \sigma_x \rangle + \langle \sigma_z \otimes \sigma_z \rangle = 2.$$

Example 2, I

- **Observation.** We show that

$$\mathrm{Tr}(AB^{T1}) = \mathrm{Tr}(A^{T1}B).$$

- *Proof.*–Remember that

$$(X^{T1})_{ij,kl} = X_{kj,il}.$$

- Based on that

$$(AB)_{ij,kl} = \sum_{m,n} A_{ij,mn}B_{mn,kl},$$

$$(AB^{T1})_{ij,kl} = \sum_{m,n} A_{ij,mn}B_{kn,ml},$$

$$\mathrm{Tr}(AB^{T1}) = \sum_{i,j}\sum_{m,n} A_{ij,mn}(B^{T1})_{mn,ij} = \sum_{i,j}\sum_{m,n} A_{ij,mn}B_{in,mj},$$

$$\mathrm{Tr}(A^{T1}B) = \sum_{i,j}\sum_{m,n}(A^{T1})_{ij,mn}B_{mn,ij}. = \sum_{i,j}\sum_{m,n} A_{mj,in}B_{mn,ij}.$$

- We can see that $\mathrm{Tr}(AB^{T1}) = \mathrm{Tr}(A^{T1}B)$, since if we exchange i and m with each other then we get one formula from the other. □

Example 2, II

- **Example 2**: Let us design an entanglement witness that detects the state $|\Psi\rangle$ as entangled. Such a witness can be defined as

$$W = |v\rangle\langle v|^{\mathrm{T1}},$$

where $|v\rangle$ is the eigenvector of $|\Psi\rangle\langle\Psi|^{\mathrm{T1}}$ with the smallest eigenvalue (which is negative).

- *Proof*. If $|v\rangle$ is the eigenvector of $|\Psi\rangle\langle\Psi|^{\mathrm{T1}}$ with the smallest eigenvalue then we have

$$|\Psi\rangle\langle\Psi|^{\mathrm{T1}}|v\rangle = \lambda|v\rangle,$$

where $\lambda < 0$. Then, we have

$$\mathrm{Tr}(W|\Psi\rangle\langle\Psi|) = \mathrm{Tr}(|v\rangle\langle v||\Psi\rangle\langle\Psi|^{\mathrm{T1}}) = \lambda < 0.$$

Here we used that $\mathrm{Tr}(A^{\mathrm{T1}}B) = \mathrm{Tr}(AB^{\mathrm{T1}})$. Thus, the witness detects the state $|\Psi\rangle$ as entangled.

Example 2, III

- Moreover, for every separable state we have

$$\mathrm{Tr}(W\varrho_{\mathrm{sep}}) = \mathrm{Tr}(|v\rangle\langle v|\varrho_{\mathrm{sep}}^{\mathrm{T1}}) > 0.$$

This can be seen knowing that $\varrho_{\mathrm{sep}}^{\mathrm{T1}} \geq 0$. $\square$

Variance based criteria

- For a bipartite system, with parties A and B, we have for both parties
$$(\Delta X_k)^2 + (\Delta Y_k)^2 \geq L_k$$
for $k = A, B$. For product states of the form $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$,we have
$$[\Delta(X_A + X_B)]^2 = \langle (X_A + X_B)^2 \rangle - \langle X_A + X_B \rangle^2 = (\Delta X_A)^2_{\Psi_A} + (\Delta X_B)^2_{\Psi_B}.$$
- This is because for product states we have
$$\langle X_A X_B \rangle - \langle X_A \rangle \langle X_B \rangle = 0.$$
Hence, we have
$$[\Delta(X_A + X_B)]^2 + [\Delta(Y_A + Y_B)]^2 \geq L_A + L_B.$$
- This is also true for separable states due to the convexity of separable states. What does this mean? The variance is concave, by mixing it will be never smaller than the average.

Variance based criteria II

- **Example**: we have

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}.$$

- Hence, using $x^2 + y^2 \geq 2xy$

$$(\Delta x)^2 + (\Delta p)^2 \geq 1.$$

- Then, we get

$$[\Delta(x_A - x_B)]^2 + [\Delta(p_A + p_B)]^2 \geq 2.$$

- The "-" sign is needed, since otherwise the relation is true for separable states, but no quantum state can violate it.

Variance based criteria III

- The entangled state
$$\sum_{k=0}^{\infty} |k\rangle|k\rangle$$
gives zero for both of the variances on the left-hand side.
- Thus, we can have
$$[\Delta(x_A - x_B)]^2 = 0$$
and
$$[\Delta(p_A + p_B)]^2 = 0.$$
- This is possible since
$$[x_A - x_B, p_A + p_B] = 0.$$

Variance based criteria IV

- **Example**: Equation with three variances

$$(\Delta j_{x,k})^2 + (\Delta j_{y,k})^2 + (\Delta j_{z,k})^2 \geq j$$

  for $k = A, B$.

- This is because

$$\begin{aligned} \langle (j_{x,k})^2 + (j_{y,k})^2 + (j_{z,k})^2 \rangle &= j(j+1), \\ \langle j_{x,k} \rangle^2 + \langle j_{y,k} \rangle^2 + \langle j_{z,k} \rangle^2 &\leq j^2. \end{aligned}$$

- Using

$$[\Delta(j_{l,A} + j_{l,B})]^2 = (\Delta j_{l,A})^2_{\Psi_A} + (\Delta j_{l,B})^2_{\Psi_B},$$

  we get for separable states

$$[\Delta(j_{x,A} + j_{x,B})]^2 + [\Delta(j_{y,A} + j_{y,B})]^2 + [\Delta(j_{z,A} + j_{z,B})]^2 \geq 2j.$$

- Any state violating this is entangled.

Variance based criteria V

- For two qubits, this relation is

$$[\Delta(j_{x,A} + j_{x,B})]^2 + [\Delta(j_{y,A} + j_{y,B})]^2 + [\Delta(j_{z,A} + j_{z,B})]^2 \geq 1.$$

- The singlet is the two-qubit state violating this criterion maximally

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B).$$

- For this state

$$[\Delta(j_{l,A} + j_{l,B})]^2 = 0$$

  for $j = x, y, z$.

- Thus, the state violates the criterion maximally.

Outline

Fully separable states/biseparable state/genuine multiparticle entanglement

- An N-qudit quantum state is called fully separable if it can be written as a convex sum of product states as

$$\varrho = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(3)}.$$

  genuine multipartite entanglement.

- An N-qudit pure quantum state is called biseparable if it can be written as the tensor product of two states as

$$|\Psi_1\rangle \otimes |\Psi_2\rangle,$$

  where $|\Psi_k\rangle$ are multiqubit states. A mixed state is called biseparable if it is the mixture of biseparable pure states.

- If a state is not biseparable then it is genuine multipartite entangled.

Fully separable states

- **Example**: fully separable state

$$|000\rangle.$$

- Biseparable state

$$\frac{1}{\sqrt{2}}|0\rangle(|00\rangle + |11\rangle).$$

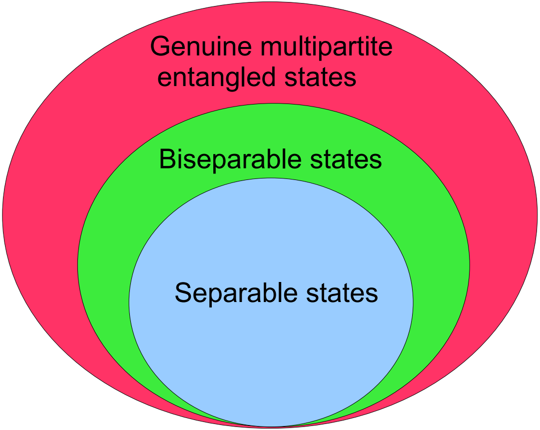
- Genuine multipartite entangled state

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

- **Example**: entanglement criterion for multipartite states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj.$$

Fully separable states



The set of entangled states and the set of separable states. Again, the set of all states is convex, similarly, as the set of separable states is convex.