Entanglement detection based on an upper bound on variances of collective observables for separable states

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Abstract

- Entanglement detection with collective measurements is important since in many experiments (e.g., optical lattices of bosonic two-state atoms) the qubits cannot be accessed individually.
- Even if the qubits can be individually accessed, measurement schemes based on collective observables are still useful since they need few measurements which is also important in experiments (e.g., see [1]).
- We present entanglement criteria, somewhat similar to the spin squeezing criterion, based on the moments or variances of the collective spin operators.
- Surprisingly, these criteria are based on an *upper* bound for variances for separable states. We present both criteria detecting entanglement in general and criteria detecting only genuine multipartite entanglement.
- Our criteria detect entanglement in the vicinity of N-qubit Dicke states with $N/2$ excitations.

Summary of related work

- Entanglement conditions based on collective measurements are built using the collective spin operators

$$J_{x/y/z} = \sum_{k=1}^{N} \sigma_{x/y/z}^{(k)}, \qquad (1)$$

where $\sigma_{x/y/z}^{(k)}$ denote Pauli spin matrices acting on qubit k.
- *Spin squeezing criterion* [2]. For separable states

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}. \qquad (2)$$

Any state violating this condition is entangled.
- *Entanglement detection around a singlet* [3]. For separable states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq 2N. \qquad (3)$$

The left hand side is minimal for many-body singlets.

Our work [4]

- Our condition: For separable states

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq N(N+1). \qquad (4)$$

For the proof see Lemma.
- For even N, the left hand side is the maximal $N(N+2)$ only for an N-qubit Dicke state with $N/2$ excitations. Such a state is the equal superposition of product states having $N/2$ ones and $N/2$ zeros.
- It can also be seen that the bound in Eq. (4) is sharp since a separable state of the from

$$|\Psi_{xy}\rangle := (|0\rangle + |1\rangle e^{i\phi})^{\otimes N}. \qquad (5)$$

for any real ϕ saturates the bound.

Lemma

- For separable states the maximum of the expression

$$a_x\langle J_x^2\rangle + a_y\langle J_y^2\rangle + a_z\langle J_z^2\rangle + b_x\langle J_x\rangle + b_y\langle J_y\rangle + b_z\langle J_z\rangle \qquad (6)$$

with $a_{x/y/z} \geq 0$ and real $b_{x/y/z}$ is the same as its maximum for translationally invariant product states (i.e., for product states of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$)
- *Proof.* When looking for the maximum of Eq. (6) for separable states, it is clearly enough to look for the maximum for pure product states.
- Let us consider a product state of the form $|\Psi\rangle = \otimes_{k=1}^{N}|\psi_k\rangle$ and use the notation $s_{x/y/z}^{(k)} := \langle\Psi|\sigma_{x/y/z}^{(k)}|\Psi\rangle$.
- We can rewrite Eq. (6) as

$$f := (a_x+a_y+a_z)N + 2\sum_{l=x,y,z} a_l \sum_{j<k} s_l^{(j)} s_l^{(k)} + b_l \sum_k s_l^{(k)}.$$

Lemma - Slide 2

- Let us look for the maximum of Eq. (7) with the constraints

$$\sum_k s_l^{(k)} = K_l$$

for $l = x,y,z$. Note that f can be written as $f = (a_x+a_y+a_z)N + a_x f_x + a_y f_y + a_z f_z$.
- Now let us first take f_x, that is, the part which depends only on the $s_x^{(k)}$ coordinates. It can be written as

$$f_x = \sum_{j<k} s_x^{(j)} s_x^{(k)} + \alpha_x \sum_k s_x^{(k)}, \qquad (7)$$

where $\alpha_x = b_x/2a_x$. We build the constraint Eq. (7) into our calculation by the substitution

$$s_x^{(N)} = K_x - \sum_{k=1}^{N-1} s_x^{(k)}. \qquad (8)$$

Lemma - Slide 3

- Thus we obtain

$$\begin{aligned} f_x &= \sum_{j<k<N} s_x^{(j)} s_x^{(k)} + \alpha_x \sum_{k=1}^{N-1} s_x^{(k)} \\ &+ (K_x - \sum_{k=1}^{N-1} s_x^{(k)})(\sum_{k=1}^{N-1} s_x^{(k)} + \alpha_x). \end{aligned}$$

Hence for any $m < N$

$$\frac{\partial f_x}{\partial s_x^{(m)}} = -s_x^{(m)} + (K_x - \sum_{k=1}^{N-1} s_x^{(k)}). \qquad (9)$$

In an extreme point this should be zero. Hence it follows that for all $m < N$ we have $s_x^{(m)} = s_x^{(N)}$, thus f_x takes its extremum for all $s_x^{(m)}$'s equal.
- Proving that the extremum is a maximum, and repeating the previous steps for f_x and f_y finish our proof.
- The proof of criterion (4) is obvious based on our Lemma.

Multipartite entanglement ...

- In a multi-qubit experiment it is important to detect genuine multi-qubit entanglement. We have to show that all the qubits were entangled with each other, not only some of them. An example of the latter case is a state of the form

$$|\Psi\rangle = |\Psi_{1..m}\rangle \otimes |\Psi_{m+1..N}\rangle \qquad (10)$$

- Note that the state given by Eq. (10) might be entangled, but it is separable with respect to the partition $(1,2,..,m)(m+1,m+2,..,N)$. Such states are called biseparable [5] and can be created from product states such that two groups of qubits do not interact.
- These concepts can be extended to mixed states. A mixed state is biseparable if it can be created by mixing biseparable pure states of the form Eq. (10). An N-qubit state is said to have genuine N-partite entanglement if it is not biseparable.

... and its detection

- For biseparable three-qubit states

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq 8 + 2\sqrt{5} \approx 12.47. \qquad (11)$$

Both the state $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ and the state $|\overline{W}\rangle = (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$ give the maximal 15 for the left-hand side of Eq. (11).
- For a four-qubit biseparable state

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq 14 + 4\sqrt{3} \approx 20.93 \qquad (12)$$

For the left hand side of Eq. (12) the maximum is 24 and it is obtained uniquely for the four-qubit Dicke state with two excitations. This state has the form $(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle)/\sqrt{6}$.
- These inequalities have recently been used for the experimental detection of multipartite entanglement [6].

Conclusions

- We have presented a method for detecting entanglement based on collective measurements. Surprisingly, it is based on an upper bound on variances of collective observables for separable states.

Related bibliography:

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