Optimal spin squeezing inequalities detect bound entanglement in spin models

Géza Tóth[1,2], Christian Knapp[3], Otfried Gühne[3], and Hans J. Briegel[3,4]

[1] ICFO-Institut de Ciències Fotòniques, E-08860 Castelldefels (Barcelona), Spain

[2] Research Institute for Solid State Physics and Optics, Hungarian Academy of Sciences, P.O. Box 49, Budapest, H-1525, Hungary

[3] Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, 6020 Innsbruck, Austria

[4] Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria

Introduction

- With the rapid development of quantum control it is now possible to create large scale entanglement in many physical systems, such as cold atoms or trapped ions.
- Entanglement conditions with collective measurements are important since in many quantum control experiments the spins cannot be individually addressed.
- We derive the complete set of such entanglement criteria for a system of spin-$\frac{1}{2}$ particles [1]. These criteria detect all entangled states that can be detected based on the first and second moments of collective angular momenta.
- When applied to several spin models, our results show the presence of bound entanglement in the thermal state.
- In particular, our criteria detect bound entanglement that has a positive partial transpose with respect to *all* bipartitions.

Motivation

- Recently, several generalized spin squeezing criteria for the detection of entanglement were derived and used even experimentally (e.g., [2-5]). These criteria detect entanglement close to various important quantum states (e.g., many-body singlet states, Dicke states, etc.) and were obtained using very different approaches.
- At this point two main questions arise:
  - Is there a systematic way of finding all such inequalities? Clearly, finding such optimal entanglement conditions is a hard task since one can expect that they contain complicated nonlinearities.
  - How strong are spin squeezing criteria? Can they detect multipartite entangled states not detectable by the PPT criterion or other *bipartite* entanglement criteria?

Spin squeezing criterion

- We call a quantum state fully separable states if it can be written as

$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes \ldots \otimes \rho_l^{(N)}, \qquad (1)$$

where $\sum_l p_l = 1$ and $p_l > 0$. Otherwise, we call the state entangled.

- The spin squeezing criterion [2] for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}, \qquad (2)$$

where $J_l := \frac{1}{2}\sum_{k=1}^N \sigma_l^{(k)}$ for $l = x, y, z$ are the collective angular momentum components and $\sigma_l^{(k)}$ are Pauli matrices. If this inequality is violated then the state is entangled.

- In practice this means that the angular momentum of the state has a small variance in one direction, while in an orthogonal direction the angular momentum is large.

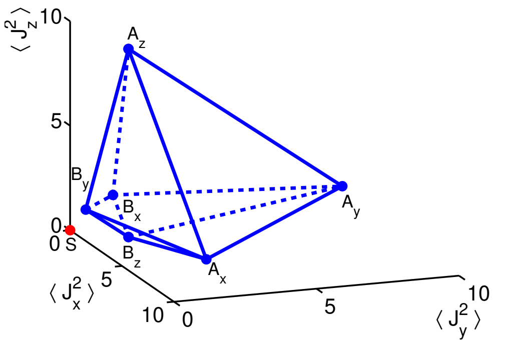
Optimal spin squeezing

- For separable states the following inequalities hold:

$$\begin{aligned} \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq N(N+2)/4, \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\ \langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\ (N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + N(N-2)/4, \end{aligned}$$

where k, l, m take all permutations of x, y, z.

- For fixed $\langle J_k \rangle$ these describe a polytope in the space of the $\langle J_k^2 \rangle$. The figure shows this polytope for $N = 6$:



The polytope

- The coordinates of the extreme points are

$$A_x \left[\frac{N^2}{4} - \kappa(\langle J_y \rangle^2 + \langle J_z \rangle^2), \frac{N}{4} + \kappa\langle J_y \rangle^2, \frac{N}{4} + \kappa\langle J_z \rangle^2\right],$$

$$B_x \left[\langle J_x \rangle^2 + \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N}, \frac{N}{4} + \kappa\langle J_y \rangle^2, \frac{N}{4} + \kappa\langle J_z \rangle^2\right],$$

where $\kappa := (N-1)/N$. Points $A_{y/z}$ and $B_{y/z}$ can be obtained from these by permuting the coordinates.

- For $\langle J_k \rangle = 0$ and even N, states corresponding to A_x and B_x are

$$\rho_{A_x} = \frac{1}{2}\left[(|+1_x\rangle\langle+1_x|)^{\otimes N} + (|-1_x\rangle\langle-1_x|)^{\otimes N}\right] \qquad (4)$$

and

$$\rho_{B_x} = (|+1_x\rangle\langle+1_x|)^{\otimes N/2} \otimes (|-1_x\rangle\langle-1_x|)^{\otimes N/2}. \qquad (5)$$

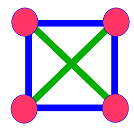
- For $\langle J_k \rangle \neq 0$ constructing such states is more complicated and is explained in Ref. [1].

Small spin clusters

- Let us consider four spin-1/2 particles, interacting via the Hamiltonian

$$H = (h_{12} + h_{23} + h_{34} + h_{41}) + J_2(h_{13} + h_{24}), \qquad (6)$$

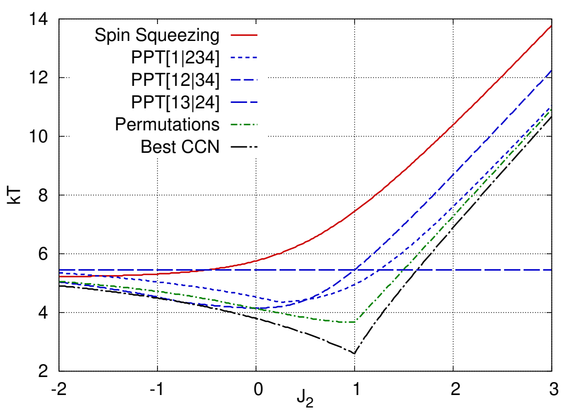
where $h_{ij} = \sigma_x^{(i)} \otimes \sigma_x^{(j)} + \sigma_y^{(i)} \otimes \sigma_y^{(j)} + \sigma_z^{(i)} \otimes \sigma_z^{(j)}$ is a Heisenberg interaction between the qubits i, j.



- Such a Hamiltonian is used to describe cuprate and polyoxovanadate clusters [6,7].
- For the above Hamiltonian we compute the thermal state $\rho(T, J_2) \propto \exp(-H/kT)$ and investigate its separability properties.
- For several separability criteria (i.e., partial transpose criterion, criteria based on symmetric extensions, computable cross norm criterion, and other permutation criteria) we calculate the maximal temperature, below which the criteria detect the thermal state as entangled.

Small spin clusters II

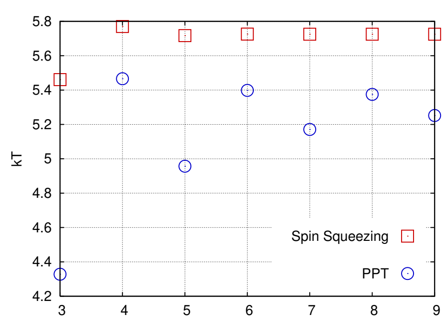
- Bound temperatures for entanglement



- For $J_2 \gtrsim -0.5$, the spin squeezing inequality is the strongest criterion for separability. It allows to detect entanglement even if the state has a positive partial transpose (PPT) with respect to all bipartitions.
- Note that multipartite bound entanglement that is PPT with respect to all partitions is very challenging to detect.

Spin chains

- We found bound entanglement that is PPT with respect to all bipartitions in XY and Heisenberg chains, and also in XY and Heisenberg models on a completely connected graph, up to 9 qubits.
- The dependence of the critical temperatures for the PPT and the optimal spin squeezing criteria as a function of the number of spins in the Heisenberg chain is the following



- Thus for these models, which appear in nature, there is a considerable temperature range in which the system is already PPT but not yet separable.

Related bibliography

1. G. Tóth, C. Knapp, O. Gühne, and H.J. Briegel, arXiv:quant-ph/0702219.
2. A. Sørensen *et al.*, Nature **409**, 63 (2001).
3. G. Tóth, PRA **69**, 052327 (2004).
4. G. Tóth, J. Opt. Soc. Am. B **24**, 275 (2007); C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. **98**, 063604 (2007).
5. J. Korbicz, J.I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **95**, 120502 (2005); J. Korbicz *et al.*, Phys. Rev. A **74**, 052319 (2006).
6. I. Bose and A. Tribedi, Phys. Rev. A **72**, 022314 (2005); T. Vértesi and E. Bene, Phys. Rev. B **73**, 134404 (2006).
7. See, e.g. R. Basler *et al.*, Inorg. Chem. **41**, 5675 (2002); J.T. Haraldsen, T. Barnes, and J.L. Musfeldt, Phys. Rev. B **71**, 064403 (2005).