Spin-squeezing inequalities for entanglement detection in cold gases

G. Tóth[1,2,3]

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary

Institute of Theoretical Physics, University of Ulm,
13 November 2013


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangel'sk
Luleå
Oulu
Umeå
Trondheim
Gulf of Bothnia
Lake Onega
Lake Ladoga
Tampere
Helsinki
Turku
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
Bergen
Oslo
Gävle
Stockholm
ÅLAND ISLANDS
Gulf of Finland
St. Petersburg
Tallinn
ESTONIA
Rockall (U.K.)
HEBRIDES
Stavanger
Göteborg
Gotland
LATVIA
Riga
North Atlantic Ocean
Aberdeen
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
UNITED KINGDOM
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Liverpool
Leeds
Manchester
Birmingham
Cardiff
London
Kaliningrad
RUSSIA
Bornholm
Minsk
Mahilyow
BELARUS
Hrodna
Homyel'
Gdańsk
Hamburg
Bremen
Berlin
Poznań
POLAND
Warsaw
Łódź
Wroclaw
Brest
Kiev
Amsterdam
NETH.
Rotterdam
Essen
Cologne
Leipzig
Bonn
GERMANY
Celtic Sea
English Channel
Brussels
Lille
BEL.
Frankfurt am Main
Prague
CZECH REPUBLIC
Brno
Kraków
Rivne
L'viv
UKRAINE
Guernsey (U.K.)
Jersey (U.K.)
LUX.
Luxembourg
Paris
Stuttgart
Strasbourg
Munich
SLOVAKIA
Bratislava
Vienna
AUSTRIA
Budapest
HUNGARY
Chernivtsi
Chisinau
Mykolayiv
MOLDOVA
Iaşi
Cluj-Napoca
ROMANIA
Nantes
FRANCE
Bay of Biscay
Zürich
Bern
Geneva
SWITZ.
LIECH.
Vaduz
Ljubljana
SLOVENIA
Zagreb
Bucharest
Lyon
A Coruña
Bordeaux
Bilbao
Milan
Venice
Turin
Genoa
CROATIA
BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
SERBIA AND MONTENEGRO
Varna
BULGARIA
Sofia
Toulouse
Andorra la Vella
Marseille
MONACO
SAN MARINO
Florence
Ligurian Sea
Adriatic Sea
Porto
ANDORRA
Zaragoza
Corsica
ITALY
Rome
Skopje
F.Y.R.O.M.
Istanbul
PORTUGAL
Madrid
Barcelona
VATICAN CITY
Tirana
ALB.
Thessaloniki
Lisbon
SPAIN
Valencia
Balearic Sea
Naples
Sardinia
Tyrrhenian Sea
GREECE
Aegean Sea
Athens
BALEARIC ISLANDS
Sevilla
Cagliari
Ionian Sea
Mediterranean Sea
Gibraltar (U.K.)
Málaga
Palermo
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
TUNISIA
Crete
Valletta
MALTA
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Why spin squeezing inequalities for $j > \frac{1}{2}$ is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Most experiments use atoms with $j > \frac{1}{2}$.

Articles reviewed in this talk

- Simple entanglement conditions for singlets

GT	PRA 2004	
GT, M.W. Mitchell	NJP 2010	(Singlets in cold gases)

- Complete set of inequalities for spin-$\frac{1}{2}$ particles

GT, C. Knapp, O. Gühne, H.J. Briegel	PRL 2007
GT, C. Knapp, O. Gühne, H.J. Briegel	PRA 2009
GT	JOSAB B 2007

- Complete set of inequalities for spin-j particles

G. Vitagliano, P. Hyllus, I.L. Egusquiza, GT	PRL 2011
G. Vitagliano, I. Apellaniz, I.L. Egusquiza, GT	arxiv 2013

Outline

Physical systems

State-of-the-art in experiments

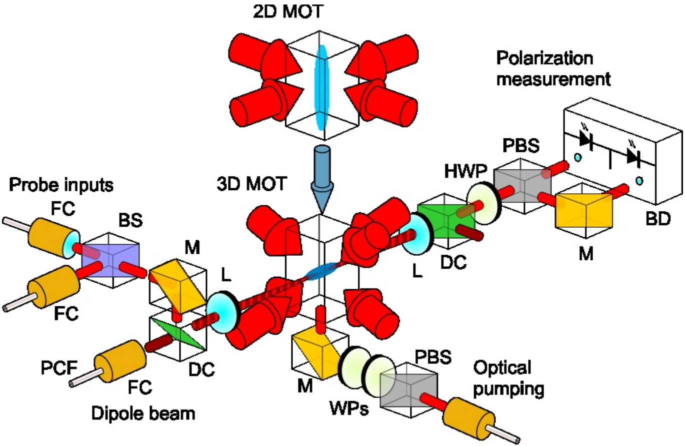
- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Physical systems II

For example: Spin squeezing in a cold atomic ensemble



Picture from M.W. Mitchell, ICFO, Barcelona.

Outline

Entanglement

Definition

A multiparticle state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not fully separable, then it is called entangled.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

The standard spin-squeezing criterion

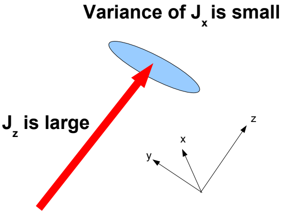
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

- If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States violating it are like this:



Outline

The inequality with three variances

- For separable states we have

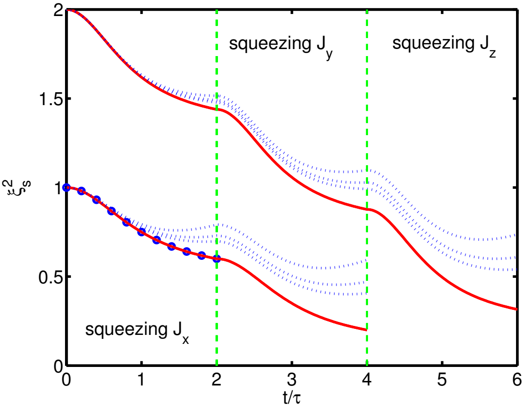
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj.$$

Any state that violates the above criterion is entangled.
[GT, Phys. Rev. A 69, 052327 (2004).]

- The left-hand side is zero for the multi-particle singlet.
- Experimental tests:
  - Photons: T.Sh. Iskhakov, I.N. Agafonov, M.V. Chekhova, G. Leuchs, PhysRevLett. 109 150502 (2012).
  - Fermions: J. Meineke, J.-P. Brantut, D. Stadler, T. Müller, H. Moritz,T. Esslinger, Nature Phys. 8, 455 (2012).

The inequality with three variances II

- Cold gas experiment proposal.
  [GT, M.W. Mitchell, New J. Phys. 12, 053007 (2010).]



- Experiments have been carried out by the Mitchell group at ICFO, Barcelona.

The inequality with three variances III

- The collective variances can be expressed with susceptibilities. We need
  - Thermal equilibrium,
  - Hamiltonians respecting certain symmetries.

[M. Wieśniak, V. Vedral, and Č. Brukner, New J. Phys. 7 258 (2005).]

- It is possible to obtain temperature limits for entanglement for real systems. For example, see
  - I. Bose and A. Tribedi, Phys. Rev. A 72, 022314 (2005),
  - T. Vértesi and E. Bene, Phys. Rev. B 73, 134404 (2006).

The inequality with three variances IV

- In the isotropic case, appears also in the structure factor based entanglement conditions.

[O. Marty, M. Epping, H. Kampermann, D. Bruss, M.B. Plenio, and M. Cramer, arXiv:1310.0929.]

- Important property: it can detect bound entangled states that have a positive partial transpose.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. A 79, 042334 (2009).]

The inequality with three variances V

- What does the amount of violation mean?

- It can be used to get a lower bound on the number of spins unentangled with the rest.

- For states of the form

$$\otimes_{n=1}^{M} |\Psi_n\rangle \otimes |\Psi_{N-M}\rangle$$

  we have

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Mj.$$

- For states that are the mixtures of pure states with at least M unentangled spins we have the same constraint.
  [GT, M.W. Mitchell, New J. Phys. 12, 053007 (2010)]

Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled.

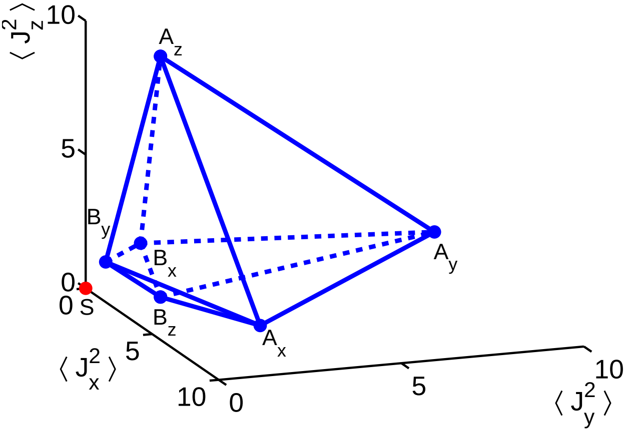
$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4}, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2}, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},
\end{aligned}$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

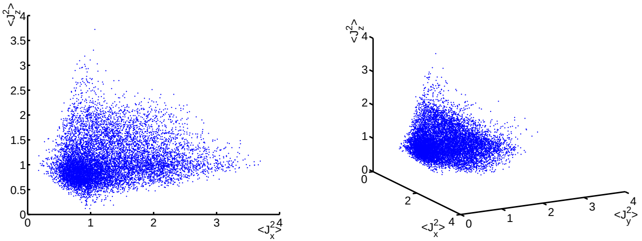
Generalized spin squeezing criteria for $j = \frac{1}{2}$

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- For $\langle \vec{J} \rangle = 0$ and $N = 6$ the polytope is the following:



Completeness

- Random separable states:

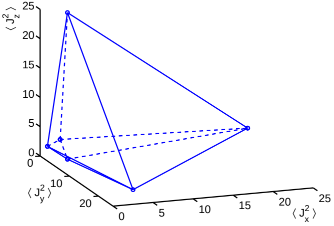


- The completeness can be proved for large *N*.

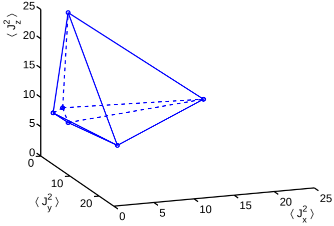
Completeness II

The polytope for $N = 10$ and

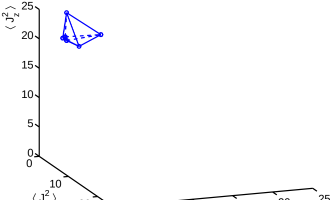
$J = (0,0,0)$,



$J = (0,0,2.5)$,



and $J = (0,0,4.5)$.



Completeness III

- Experimental tests: on-going experiments in the group of Carsten Klempt (Hannover) to detect Dicke states in cold gases.

- Other uses: one can obtain analytically

$$\min_{\Psi \text{ separable}} \sum_{\alpha=x,y,z} m_\alpha (\Delta J_\alpha)^2 \tag{2}$$

appearing in structure factor based entanglement detection.

[O. Marty, M. Epping, H. Kampermann, D. Bruss, M.B. Plenio, and M. Cramer, arXiv:1310.0929.]

Outline

"Modified" quantities for $j > \frac{1}{2}$

- For the $j = \frac{1}{2}$ case, the SSIs were developed based on the first and second moments and variances of the such collective operators.

- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{J}_k^2 \rangle := \langle J_k^2 \rangle - \langle \sum_n (j_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle j_k^{(n)} j_k^{(m)} \rangle$$

  and the modified variance

$$(\tilde{\Delta} J_k)^2 := (\Delta J_k)^2 - \langle \sum_n (j_k^{(n)})^2 \rangle.$$

- These are essential to get tight equations for $j > \frac{1}{2}$.

The inequalities for $j>\frac{1}{2}$ with the angular momentum coordinates

- For fully separable states of spin-j particles, all the following inequalities are fulfilled

$$
\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq Nj(Nj+1), \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj, \\
\langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N-1)j^2 &\leq (N-1)(\tilde{\Delta} J_m)^2, \\
(N-1)\left[(\tilde{\Delta} J_k)^2 + (\tilde{\Delta} J_l)^2\right] &\geq \langle \tilde{J}_m^2 \rangle - N(N-1)j^2,
\end{aligned}
$$

  where k, l, m take all possible permutations of x, y, z.

- Violation of any of the inequalities implies entanglement.

Completeness

- In the large N limit, the inequalities with the angular momentum are complete.
- It is not possible to find new entanglement conditions based on $\langle J_k \rangle$ and $\langle \tilde{J}_k^2 \rangle$ that detect more states.

Outline

The usual spin squeezing inequality for $j > \frac{1}{2}$

- The standard spin-squeezing inequality becomes

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} + \frac{\sum_n (j^2 - \langle (j_x^{(n)})^2\rangle)}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

  Violated only if there is entanglement between the spin-j particles.

- The second term on the LHS is nonnegative.

Outline

The inequalities for $j > \frac{1}{2}$ with the G_k's

- Collective operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

where $l = 1, 2, ..., d^2 - 1$ and $g_l^{(k)}$ are the SU(d) generators.

- We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.

The inequalities for $j > \frac{1}{2}$ with the G_k's

- The SSIs for G_k have the general form

$$(N-1)\sum_{k\in I}(\tilde{\Delta} G_k)^2 - \sum_{k\notin I}\langle(\tilde{G}_k)^2\rangle \geq -2N(N-1)\frac{(d-1)}{d}.$$

- For instance, for the $d = 3$ case, the SU(d) generators can be the eight Gell-Mann matrices.
- I is a subset of indices between 1 and M. We have 2^M equations!

Outline

An example: The criterion for SU(d) singlets

A condition for two-producibility (i.e., a higher form of entanglement) for N qudits of dimension d is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Spin squeezing inequalities for arbitrary spin, PRL 2011.]

Group

Philipp Hyllus	Research Fellow (2011-2012)
Zoltán Zimborás	Research Fellow (2012-)
Iñigo Urizar-Lanz	Ph.D. Student
Giuseppe Vitagliano	Ph.D. Student
Iagoba Apellaniz	Ph.D. Student

- Topics
  - Multipartite entanglement and its detection
  - Metrology, cold gases
  - Collaborating on experiments:
    - Weinfurter group, Munich, (photons)
    - Mitchell group, Barcelona, (cold gases)
- Funding:
  - European Research Council starting grant 2011-2016, 1.3 million euros.
  - CHIST-ERA QUASAR collaborative EU project.
  - Grants of the Spanish Government and the Basque Government

Summary

- Full set of generalized spin squeezing inequalities with J_l with $l = x, y, z$ for $j > \frac{1}{2}$.
- Large set of inequalities with the other collective operators.
- These might make possible new experiments and make existing experiments simpler.

See: G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Phys. Rev. Lett. 107, 240502 (2011) + arxiv:1310.2269.

See `www.gtoth.eu` for the slides

THANK YOU FOR YOUR ATTENTION!

erc European Research Council
chist-era
MINISTERIO DE CIENCIA E INNOVACIÓN
EUSKO JAURLARITZA GOBIERNO VASCO