

Permutationally invariant quantum tomography

G. Tóth^{1,2,3}, W. Wieczorek^{4,5}, D. Gross⁶, R. Krischek^{4,5},
C. Schwemmer^{4,5}, and H. Weinfurter^{4,5}

¹Theoretical Physics, The University of the Basque Country, Bilbao, Spain

²IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

³Research Institute for Solid State Physics and Optics, Budapest, Hungary

⁴Max-Planck-Institut für Quantenoptik, Garching, Germany

⁵Department für Physik, Ludwig-Maximilians-Universität, München, Germany

⁶Institute for Theoretical Physics, Leibniz University Hannover, Hannover, Germany

Stockholm, October 5, 2010



1 Motivation

- Why quantum tomography is important?

2 Quantum experiments with multi-qubit systems

- Physical systems
- Local measurements

3 Full quantum state tomography

- Basic ideas and scaling
- Experiments

4 Permutationally invariant tomography

- Main results
- Example: 4-qubit Dicke state, optimized settings

5 Extra slide 1: Number of settings

Why tomography is important?

- Many experiments aiming to create many-body entangled states
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales **exponentially** with the number of qubits.

- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - Main results
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

State-of-the-art in experiments

- 14 qubits with trapped cold ions
T. Monz, P. Schindler, J.T. Barreiro, M. Chwalla, D. Nigg, W.A. Coish, M. Harlander, W. Haensel, M. Hennrich, R. Blatt, arxiv:1009.6126, 2010.
- 10 qubits with photons
W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, J.-W. Pan, Nature Physics, 6, 331 (2010).

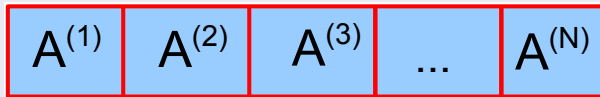
- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - Main results
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

Only local measurements are possible

Definition

A single **local measurement setting** is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.



- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)} \rangle, \langle A^{(1)}A^{(3)} \rangle, \langle A^{(1)}A^{(2)}A^{(3)} \rangle, \dots$$

- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - Main results
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

Full quantum state tomography

- The density matrix can be reconstructed from 3^N measurement settings.

Example

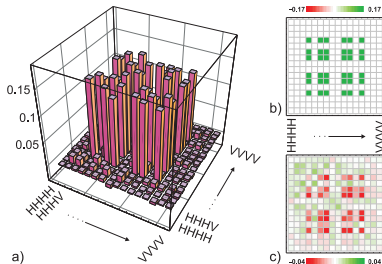
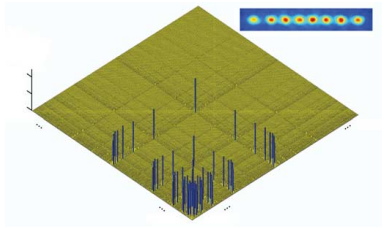
For $N = 4$, the measurements are

1.	X	X	X	X
2.	X	X	X	Y
3.	X	X	X	Z
		...		
3^4 .	Z	Z	Z	Z

- Note again that the number of measurements scales **exponentially** in N .

- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - Main results
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

Experiments with ions and photons



- H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, *Nature* 438, 643-646 (2005).
- N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, *Phys. Rev. Lett.* 98, 063604 (2007).

Approaches to solve the scalability problems

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.
S.T. Flammia et al., [arxiv:1002.3839](#); M. Cramer, M.B. Plenio, [arxiv:1002.3780](#).
- If the state is of low rank, we need fewer measurements.
D. Gross et al., [arxiv:0909.3304](#).
- We make tomography in a subspace of the density matrices (our approach)

- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - **Main results**
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

Permutationally invariant part of the density matrix

Permutationally invariant part of the density matrix:

$$\rho_{\text{PI}} = \frac{1}{N!} \sum \Pi_k \rho \Pi_k^\dagger,$$

where Π_k are all the permutations of the qubits.

- Related literature: Reconstructing ρ_{PI} for spin systems.
[G. M. D'Ariano *et al.*, *J. Opt. B* **5**, 77 (2003).]
- Photons in a single mode optical fiber are always in a permutationally invariant state. Small set of measurements are needed for their characterization (experiments).
[R.B.A. Adamson *et al.*, *Phys. Rev. Lett.* **98**, 043601 (2007); R.B.A. Adamson *et al.*, *Phys. Rev. A* 2008; L. K. Shalm *et al.*, *Nature* **457**, 67 (2009).]

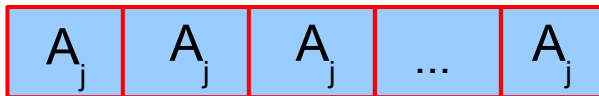
Main results

Features of our method:

- 1 Is for **spatially separated qubits**.
- 2 Needs the **minimal number of measurement settings**.
- 3 Uses the measurements that lead to the **smallest uncertainty possible** of the elements of ρ_{PI} .
- 4 **Gives an uncertainty** for the recovered expectation values and density matrix elements.
- 5 If ρ_{PI} is entangled, so is ρ . Can be used for entanglement detection!

Measurements

- We measure the same observable A_j on all qubits. (Necessary for optimality.)



- Each qubit observable is defined by the measurement directions \vec{a}_j using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$\mathcal{D}_N = \binom{N+2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$

What do we get from the measurements?

We obtain the expectation values for

$$\langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{PI} \rangle$$

for $j = 1, 2, \dots, \mathcal{D}_N$ and $n = 0, 1, \dots, N$.

How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

$$\underbrace{\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \rangle}_{\text{Bloch vector elements}} = \sum_{j=1}^{\mathcal{D}_N} \underbrace{c_j^{(k,l,m)}}_{\text{coefficients}} \times \underbrace{\langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \rangle}_{\text{Measured data}}.$$

- Coefficients are not unique if $n > 0$.

Uncertainties

The uncertainty of the reconstructed Bloch vector element is

$$\mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}}] = \sum_{j=1}^{\mathcal{D}_N} |c_j^{(k,l,m)}|^2 \mathcal{E}^2[(A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\text{PI}}].$$

- For a fixed set of A_j , we have a formula to find the $c_j^{(k,l,m)}$'s giving the minimal uncertainty.

Optimization for A_j

- We have to find \mathcal{D}_N measurement directions \vec{a}_j on the Bloch sphere minimizing the variance

$$(\mathcal{E}_{\text{total}})^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \right] \times \left(\frac{N!}{k!l!m!n!} \right).$$

Summary of algorithm

Obtaining the "total uncertainty" for given measurements

$$\left. \begin{array}{l} \rho_0, \text{ the state we expect} \\ A_j, \text{ what we measure} \end{array} \right\} \Rightarrow \text{BOX \#1} \Rightarrow (\mathcal{E}_{\text{total}})^2$$

Evaluating the experimental results

$$\left. \begin{array}{l} \text{measurement results} \\ A_j \end{array} \right\} \Rightarrow \text{BOX \#2} \Rightarrow \left\{ \begin{array}{l} \text{Bloch vector elements} \\ \text{variances} \end{array} \right.$$

How much is the information loss?

Estimation of the fidelity $F(\rho, \rho_{\text{PI}})$:

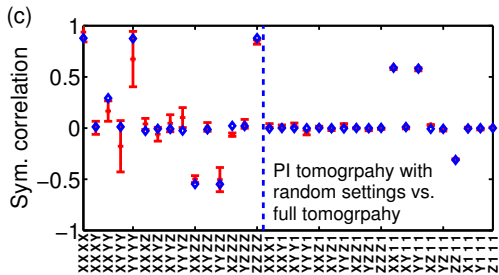
$$F(\rho, \rho_{\text{PI}}) \geq \langle P_s \rangle_{\rho}^2 \equiv \langle P_s \rangle_{\rho_{\text{PI}}}^2,$$

where P_s is the projector to the N -qubit symmetric subspace.

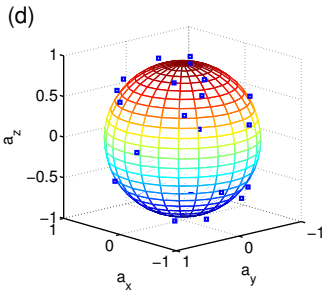
- $F(\rho, \rho_{\text{PI}})$ can be estimated only from ρ_{PI} !

- 1 **Motivation**
 - Why quantum tomography is important?
- 2 **Quantum experiments with multi-qubit systems**
 - Physical systems
 - Local measurements
- 3 **Full quantum state tomography**
 - Basic ideas and scaling
 - Experiments
- 4 **Permutationally invariant tomography**
 - Main results
 - Example: 4-qubit Dicke state, optimized settings
- 5 **Extra slide 1: Number of settings**

Random settings (exp.)

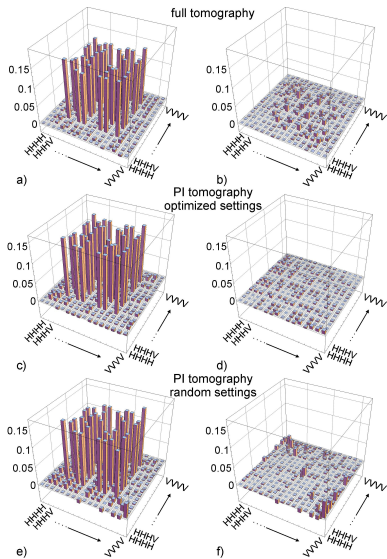


The measured correlations



\vec{a}_j measurement directions

Density matrices (exp.)

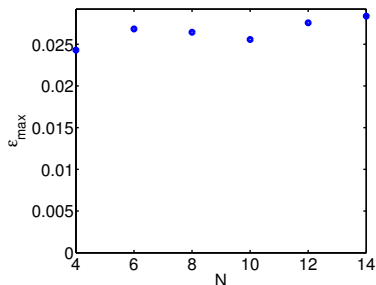


PI tomography for larger systems

- We determined the optimal A_j for p.i. tomography for $N = 4, 6, \dots, 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon_{\max}^2 = \max_{k,l,m,n} \mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}}]$$

(Total count is the same as in the experiment: 2050.)



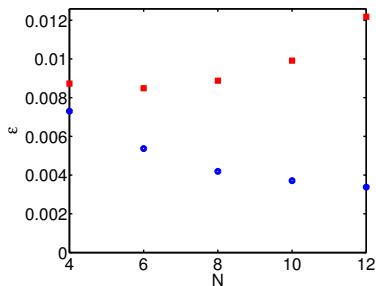
Expectation values directly from measured data

- Operator expectation values can be recovered directly from the measurement data as

$$\langle Op \rangle = \sum_{j=1}^{\mathcal{D}_N} \sum_{n=1}^N c_{j,n}^{Op} \langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{PI} \rangle,$$

where the $c_{j,n}^{Op}$ are constants.

- $Op = |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|$, blue: $\varrho_0 \propto \mathbb{1}$, red: upper bound for any ϱ_0 .



Summary

- We discussed permutationally invariant tomography for large multi-qubits systems.
- It paves the way for quantum experiments with more than 6 – 8 qubits.

See:

G. Tóth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Permutationally invariant quantum tomography, arxiv:1005.3313.

THANK YOU FOR YOUR ATTENTION!

How many settings we need?

- Expectation values of $(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{PI}$ are needed.
- For a given n , the dimension of this subspace is $\mathcal{D}_{(N-n)}$ (simple counting).
- Operators with different n are orthogonal to each other.
- Every measurement setting gives a single real degree of freedom for each subspace
- Hence the number of settings cannot be smaller than the largest dimension, which is \mathcal{D}_N .