Detecting k-particle entanglement with spin squeezing inequalities (a derivation from arxiv:1104.3147, ☛ talk by G. Vitagliano)

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Norwegian Sea
Barents Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangelsk
Luleå
Oulu
Umeå
Lake Onega
Lake Ladoga
Trondheim
Gulf of Bothnia
Tampere
Helsinki
Turku
Gulf of Finland
St. Petersburg
Gävle
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
Bergen
Oslo
Stockholm
ALAND ISLANDS
Tallinn
ESTONIA
Rockall (U.K.)
HEBRIDES
Stavanger
Göteborg
Gotland
LATVIA
Riga
North Atlantic Ocean
Aberdeen
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
UNITED KINGDOM
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Liverpool
Leeds
Manchester
Birmingham
Cardiff
London
Kaliningrad
RUSSIA
Bornholm
Gdańsk
Minsk
Mahilyow
BELARUS
Hrodna
Homyel'
Hamburg
Bremen
Berlin
Poznań
POLAND
Warsaw
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Wrocław
Brest
Kiev
Amsterdam
NETH.
Rotterdam
Essen
Cologne
Leipzig
GERMANY
Rivne
UKRAINE
Celtic Sea
English Channel
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Jersey (U.K.)
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Venice
Turin
Genoa
SLOVENIA
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Zagreb
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BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
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Bucharest
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Varna
BULGARIA
Sofia
Istanbul
SAN MARINO
Florence
Ligurian Sea
Adriatic Sea
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SPAIN
Valencia
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Balearic Sea
BALEARIC ISLANDS
Sardinia
Tyrrhenian Sea
Naples
GREECE
Aegean Sea
Athens
Sevilla
Gibraltar (U.K.)
Málaga
Strait of Gibraltar
Alborán Sea
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Mediterranean Sea
Algiers
Cagliari
Palermo
Sicily
Tunis
TUNISIA
Ionian Sea
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Why k-particle entanglement is important?

- Many experiments are aiming to create many-body entangled states.
- It is not sufficient to say “entangled”. We have to say something like “genuine multipartite entangled”.
- In experiments with a million atoms, we can only measure collective quantities.

See also
[L.-M. Duan, *Entanglement detection in the vicinity of arbitrary Dicke states,* arXiv:1107.5162],
[A. Sorensen and K. Molmer, *Entanglement and Extreme Spin Squeezing,* Phys. Rev. Lett. 86, 4431 (2001)].

Genuine multipartite entanglement

Definition

A state is (fully) separable if it can be written as
$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

k-producibility/k-entanglement

Definition

A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most k qubits. A mixed state is k-producible, if it is a mixture of k-producible pure states.
[O. Gühne and G. Tóth, New J. Phys 2005.]

- In many-particle systems where only collective quantities can be detected, this is the only meaningful characterization of entanglement.
- That is, genuine multipartite entanglement is very difficult to detect in such systems.

Outline

Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

Many-particle systems for j>1/2

- For spin-j particles for $j > 1/2$, we can measure the collective angular momentum operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

where $l = 1, 2, ..., d^2 - 1$ and $g_l^{(k)}$ are the SU(d) generators.

- We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.

Only collective measurements are possible

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin,
arXiv:1104.3147.]

Maximally violating states

- For $N = d$, the multipartite singlet state maximally violates the condition with $\sum_k (\Delta G_k)^2 = 0$.
- For $N < d$, there is no quantum states for which $\sum_k (\Delta G_k)^2 = 0$.
- This can be seen as follows. It is not possible to create a completely antisymmetric state of d-state particles with less than d particles.

Maximally violating states II

- A more detailed proof: For the sum of the squares of G_k we obtain

$$\sum_k (G_k)^2 = \sum_k \sum_n (g_k^{(n)})^2 + \sum_k \sum_{n\neq m} g_k^{(m)} g_k^{(n)}$$
$$= 2N\frac{d^2-1}{d}\mathbb{1} + \sum_{n\neq m} 2\left(F_{mn} - \frac{\mathbb{1}}{d}\right).$$

- Based on this and using $\langle F_{mn}\rangle \geq -1$, we can write

$$\sum_k \langle (G_k)^2\rangle \geq \frac{2N}{d}(d+1)(d-N).$$

- The bound on the right-hand side cannot be zero if $N < d$.
- For $N = d$, the sum $\sum_k \langle (G_k)^2\rangle$ is zero for the totally antisymmetric state for which $\langle F_{mn}\rangle = -1$ for all m, n.

Maximally violating states III

It can also be proved that

$$\sum_k \langle G_k^2 \rangle = 0 \quad \Leftrightarrow \quad \sum_k (\Delta G_k)^2 = 0.$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Optimal spin squeezing inequalities for arbitrary spin, arXiv:1104.3147.]

Two-producibility

- We look for the minimum of

$$\sum_k (\Delta G_k)^2 = \sum_k \langle G_k^2 \rangle - \sum_k \langle G_k \rangle^2.$$

- Let us see a two-particle system. We will compute the minimum/maximum for both terms.

First term

- First, let us see
$$\sum_k \langle G_k^2 \rangle.$$
- We have to consider symmetric and antisymmetric states. The inequality is saturated for symmetric states.

First term II

- What do we have for antisymmetric states?

$$\sum_k \langle G_k^2 \rangle = \sum_k \langle (g_k^{(1)})^2 \rangle + \sum_k \langle (g_k^{(2)})^2 \rangle + 2 \sum_k \langle (g_k^{(1)})(g_k^{(2)}) \rangle.$$

Here

$$\langle \sum_k (g_k^{(1)})^2 \rangle = \langle \sum_k (g_k^{(2)})^2 \rangle = 2(d+1)(1-1/d).$$

And,

$$\langle \sum_k (g_k^{(1)})(g_k^{(2)}) \rangle = -2(1+1/d).$$

(This is because with the flip operator F we can be write as $\sum_k g_k^{(1)} g_k^{(2)} = 2F - \frac{2}{d}$.)

- Then, we obtain

$$\sum_k \langle G_k^2 \rangle = 4(d+1)(1-2/d).$$

Second term

- Then, one has to deal with $\sum_k \langle G_k \rangle^2$. For that, we get
$$\sum_k \langle G_k \rangle^2 = \sum_k \langle g_k^{(1)} + g_k^{(2)} \rangle^2 = \sum_k \langle g_k^{(1)} \rangle^2 + \sum_k \langle g_k^{(2)} \rangle^2 + 2M,$$
where
$$M = \sum_k \langle g_k^{(1)} \rangle \langle g_k^{(2)} \rangle.$$
- Knowing that
$$\sum_k \langle g_k^{(n)} \rangle^2 \leq 2(1 - 1/d).$$
and using the Cauchy-Schwarz inequality one gets
$$\sum_k \langle G_k \rangle^2 \leq 8(1 - 1/d).$$
- Now, we have to use again that for a single qudit
$$\sum_k \langle g_k^{(n)} \rangle^2 = 2\mathrm{Tr}(\varrho^2) - 2/d.$$

Second term II

Lemma.

For bipartite antisymmetric states we have

$$Tr(\varrho_{\rm red}^2) \leq \frac{1}{2}.$$

- **Proof.** All pure two-qudit antisymmetric states can be written in some basis as

$$\alpha_{12}|\Psi_{12}^-\rangle + \alpha_{34}|\Psi_{34}^-\rangle + \alpha_{56}|\Psi_{56}^-\rangle + ...,$$

where α_{nm} are constants and

$$\Psi_{mn}^- = (|mn\rangle - |nm\rangle)/\sqrt{2}.$$

[J. Schliemann *et al.*, Phys. Rev. A **64**, 022303 (2001).]

Second Term II

- Then for the collective operators for antisymmetric states we have

$$\sum_k \langle G_k \rangle^2 \leq 4(1 - 2/d) = 4 - 8/d.$$

Symmetric and antisymmetric states

- Hence, for antisymmetric states, one gets

$$\sum_k (\Delta G_k)^2 \quad \geq \quad 4d(1-2/d) = 4(d-2) = 4d-8.$$

- For symmetric states, we get

$$\sum_k (\Delta G_k)^2 \geq 4(1-\frac{1}{d})(2+d) - 8(1-1/d) = 8d(1-\frac{1}{d}) = 8d-8.$$

This bound is always larger than the one for antisymmetric states.

Lemma

Lemma. We know that

$$\sum_k (\Delta G_k)^2_{\varrho'} = \sum_k (\Delta G_k)^2_{\varrho}.$$

where

$$\varrho' = P_a \varrho P_a + P_s \varrho P_s.$$

It is the same as

$$\varrho' = \frac{1}{2}(\varrho + F\varrho F),$$

where F is the flip operator. Hence, the coherences between the symmetric and asymmetric parts need not be considered.

Proof. The variance of a collective operator is permutationally invariant.

The criterion

A condition for two-producibility for N qudits of dimension d is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

Summary

- We showed that a certain generalized spin-squeezing inequality can be used to detect three-particle entanglement.
- The inequality detects states close to many-body singlet states.

See:
G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin,
arXiv:1104.3147.

THANK YOU FOR YOUR ATTENTION!

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