Entanglement detection close to multi-qubit Dicke states in photonic experiments (review)

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Based on the following papers:
GT, JOSAB 2007
GT, C. Knapp, O. Gühne, H. Briegel, PRL 2007
W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009
GT, M.W. Mitchell, NJP 2010
GT, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, PRL 2010
G. Vitagliano, P. Hyllus, I. Egusquiza, GT, PRL 2011,
I. Urizar-Lanz, P. Hyllus, I. Egusquiza, GT, arxiv 2012; etc.

Mátraháza Workshop, Hungary, May 2012


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland
(DENMARK)
Jan Mayen
(NORWAY)
Greenland
Sea
Denmark
Strait
Barents
Sea
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangel
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
Luleå
Oulu
Umeå
FINLAND
Lake
Onega
Trondheim
SWEDEN
Gulf
of
Bothnia
Tampere
Lake
Ladoga
Tórshavn
Faroe Islands
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Turku
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TURKEY
GREECE
Aegean
Sea
Sardinia
BALEARIC
ISLANDS
Athens
Sevilla
Cagliari
Ionian
Sea
Mediterranean Sea
Gibraltar
(U.K.)
Málaga
Palermo
Sicily
Strait of Gibraltar
Ceuta
(SPAIN)
Alborán
Sea
Melilla
(SPAIN)
Oran
Algiers
Tunis
Scale 1: 19,500,000
Lambert Conformal Conic Projection,
standard parallels 40°N and 56°N
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
300 Kilometers
300 Miles
Boundary representation
not necessarily authoritative


Outline

Why Dicke states are important?

- They naturally arise in symmetric systems.
- They are highly entangled and possess high level of multipartite entanglement.
- They are very useful in metrology.

Dicke states

- We will consider mostly symmetric Dicke states defined as

$$|\mathcal{D}_N^{(n)}\rangle = \binom{n}{N}^{-\frac{1}{2}} \sum_k \mathcal{P}_k(|1\rangle^{\otimes n}|0\rangle^{\otimes(N-n)}),$$

  where the summation is over all different permutations.

- For even N, $|\mathcal{D}_N^{(n)}\rangle$ is the Dicke state with $\langle J_z\rangle = 0$. It appears also in Dicke's original paper as the most superradiant state.

- Another interesting state is the permutationally invariant singlet

$$\varrho_s \propto \sum_\alpha |j=0, j_z=0, \alpha\rangle\langle j=0, j_z=0, \alpha|.$$

Outline

Genuine multipartite entanglement

Definition

A state is (fully) separable if it can be written as $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

Genuine multipartite entanglement II

- The following 10-particle state is entangled

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle^{\otimes 8}$$

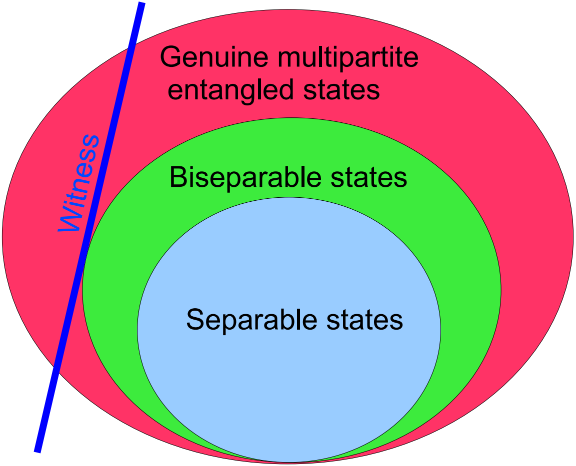
  but it is not 10-particle multipartite entangled.

- We can see only two entangled particles.

- For a N-qubit experiment, we need that all qubits are entangled with each other.

Convex sets for the multipartite case

- The idea of convex sets also work for the multi-qubit case: A state is biseparable if it can be obtained by mixing pure biseparable states.



Outline

Entanglement close to symmetric Dicke states

- A state is genuine multipartite entangled if

$$\mathrm{Tr}(|\mathcal{D}_N^{(N/2)}\rangle\langle\mathcal{D}_N^{(N/2)}|\varrho) > \frac{1}{2}\frac{N}{N-1}.$$

  The bound is close to $\frac{1}{2}$ for large N.
  [GT, JOSAB 2007]

- Only GHZ states and cluster states are known to have similar characteristics. Thus, symmetric Dicke states are suitable for creating genuine multipartite entanglement in many-body experiments.

- For other states, a larger fidelity is required that makes experiments more demanding. Eg., a state is entangled if

$$\mathrm{Tr}(|\mathcal{D}_N^{(1)}\rangle\langle\mathcal{D}_N^{(1)}|\varrho) > \frac{N-1}{N}.$$

Outline

Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

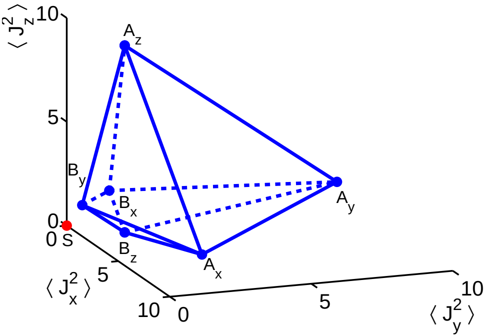
- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq N(N+2)/4, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + N(N-2)/4.
\end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.
[GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



Detecting Dicke states

- Entanglement detection close to Dicke states. For separable states,

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - N/2 \leq (N-1)(\Delta J_z)^2.$$

- The Dicke state $|\mathcal{D}_N^{(N/2)}\rangle$ has

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right),$$
$$(\Delta J_z)^2 = 0.$$

  Thus, it maximally violates the inequality.

- Detection of m-partite entanglement close to Dicke states. Such entanglement is present if

$$\frac{\langle J_x^2 + J_y^2 \rangle}{N(1/4 + (\Delta J_z)^2)} - 1 > m.$$

  [L.-M. Duan, Phys. Rev. Lett. 2011.]

Outline

“Modified” quantities for $j > \frac{1}{2}$

- For the $j = \frac{1}{2}$ case, the spin squeezing inequalities were developed based on the first and second moments and variances of the such collective operators.

- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{A}_k^2 \rangle := \langle A_k^2 \rangle - \langle \sum_n (a_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle a_k^{(n)} a_k^{(m)} \rangle$$

and the modified variance

$$(\tilde{\Delta} A_k)^2 := (\Delta A_k)^2 - \langle \sum_n (a_k^{(n)})^2 \rangle.$$

The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates II

- For fully separable states of spin-j particles, all the following inequalities are fulfilled

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq Nj(Nj+1), \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj, \\
\langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N-1)j^2 &\leq (N-1)(\tilde{\Delta} J_m)^2, \\
(N-1)\left[(\tilde{\Delta} J_k)^2 + (\tilde{\Delta} J_l)^2\right] &\geq \langle \tilde{J}_m^2 \rangle - N(N-1)j^2,
\end{aligned}$$

  where k, l, m take all possible permutations of x, y, z.

- Violation of any of the inequalities implies entanglement.

Dicke states for j>1/2

- Entanglement condition for Dicke states

$$\langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N-1)j^2 \leq (N-1)(\tilde{\Delta} J_m)^2.$$

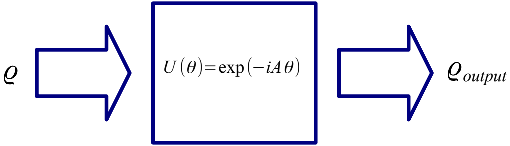
- A state maximally violating the inequality is

$$|\mathcal{D}_N^{(n,j)}\rangle = \binom{n}{N}^{-\frac{1}{2}} \sum_k \mathcal{P}_k (|+j\rangle^{\otimes N} |-j\rangle^{\otimes (N-n)}).$$

Outline

Symmetric Dicke states with $\langle J_z \rangle = 0$ are optimal for metrology

- The small parameter θ must be estimated by making measurements on the output state :



- Cramér-Rao bound with the quantum Fisher information F_Q is

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q^{\mathrm{usual}}[\varrho, A]}}.$$

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982).]

Outline

Properties of the Quantum Fisher information

For obtaining entanglement conditions, it is sufficient to know that following two relations.

1. For a pure state ϱ, we have $F[\varrho, J_l] = 4(\Delta J_l)^2_\varrho$.
2. $F[\varrho, J_l]$ is convex in the state, that is $F[p_1\varrho_1 + p_2\varrho_2, J_l] \le p_1 F[\varrho_1, J_l] + p_2 F[\varrho_2, J_l]$.

From these two statements, it also follows that $F[\varrho, J_l] \le 4(\Delta J_l)^2_\varrho$.

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982);
S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72, 3439 (1994); L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Quantum Fisher information and entanglement I

***Observation* 1**

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$F_Q[\varrho, J_l] \leq N.$$

- The GHZ state maximally violates this with N^2.
- The $|\mathcal{D}_N^{(N/2)}\rangle$ state gives roughly half of the maximal value, still very good.

[L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Quantum Fisher information and entanglement II

Observation 2

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq 2N.$$

- Both the GHZ state and the $|\mathcal{D}_N^{(N/2)}\rangle$ state maximally violate it with $N(N+2)$.

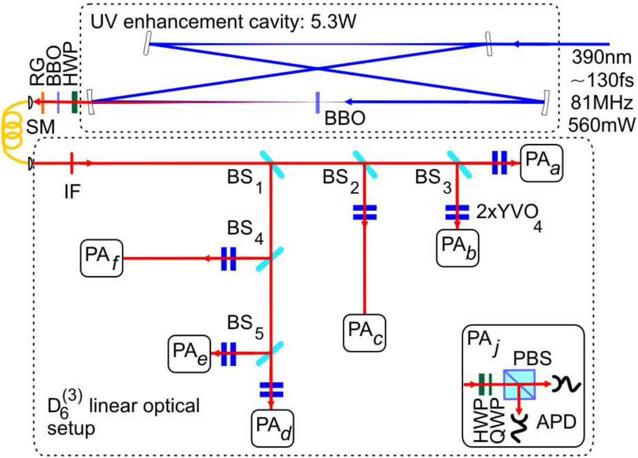
[GT, Phys. Rev. A (2012); P. Hyllus et al., Phys. Rev. A 85, 022321 (2012).]

Dicke states and metrology

- In summary, when a single metrological tasks is considered, $|\mathcal{D}_N^{(N/2)}\rangle$ is close to optimal.
- When three metrological tasks are considered, $|\mathcal{D}_N^{(N/2)}\rangle$ is optimal.

Outline

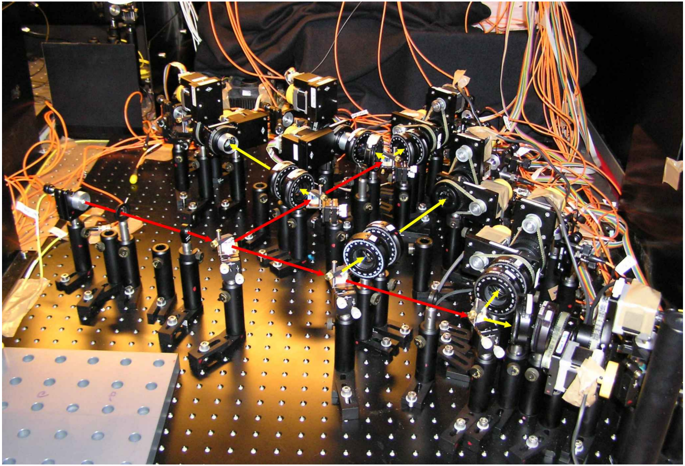
An experiment: Dicke state with photons



[W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009]

An experiment: Dicke state with photons II

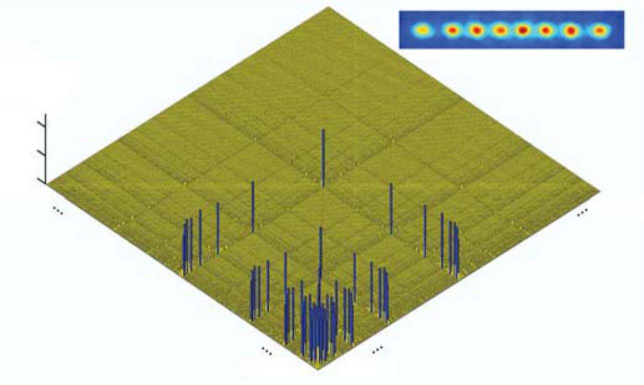
A photo of a real experiment (six-photon Dicke state, Weinfurter group, 2009):



Outline

Experiment: W-state with ions

- Experimental observation of an 8-qubit W-state with trapped ions.



H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).

Outline

Singlet states

- Another interesting state is the permutationally invariant singlet

$$\varrho_s \propto \sum_\alpha |j=0, j_z=0, \alpha\rangle\langle j=0, j_z=0, \alpha|.$$

- Entanglement condition for singlets. For separable states we have

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj.$$

[GT PRA 2004; GT and M.W. Mitchell, NJP 2010.]
[Singlet of a two-mode:two-mode system: T.Sh. Iskhakov, I.N. Agafonov, M.V. Chekhova, and G. Leuchs, arxiv:1111.2073.]
[Singlet with fermions: J. Meineke, J.-P. Brantut, D. Stadler, T. Müller, H. Moritz & T. Esslinger, Nature Phys. (2012)]

Singlet states II

- For $j = 1/2$, there is a unique permutationally invariant singlet.
  [I. Urizar-Lanz, P. Hillus, I. Egusquiza, GT arxiv 2012.]

- For a given even number of particles N, the permutationally invariant singlet state can be expressed as

$$\varrho_{\rm s} = \lim_{T\to 0} \frac{e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}}{\mathrm{Tr}\left(e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}\right)}$$

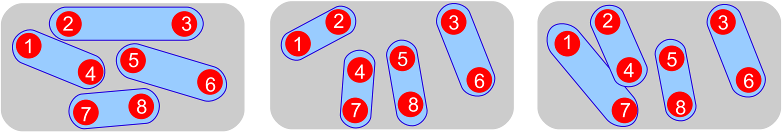
  and

$$\varrho_{\rm s} \propto \sum_k \Pi_k \left(|\Psi^-\rangle\langle\Psi^-| \otimes \cdots \otimes |\Psi^-\rangle\langle\Psi^-|\right) \Pi_k^\dagger,$$

  where the two-particle singlet is $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).$

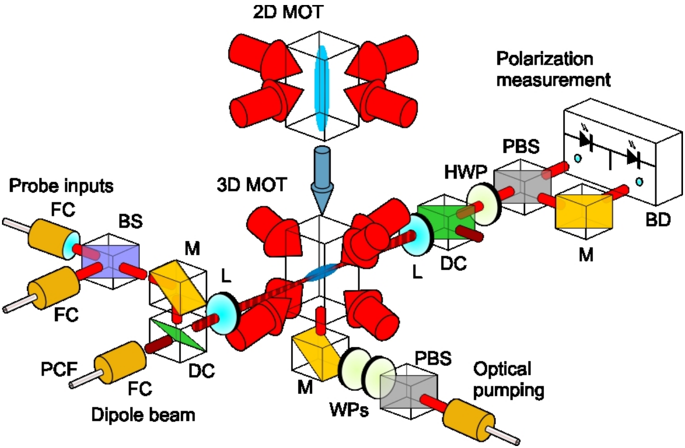
Singlet states III

- Equal mixture of all chains of two-particle singlets:



Outline

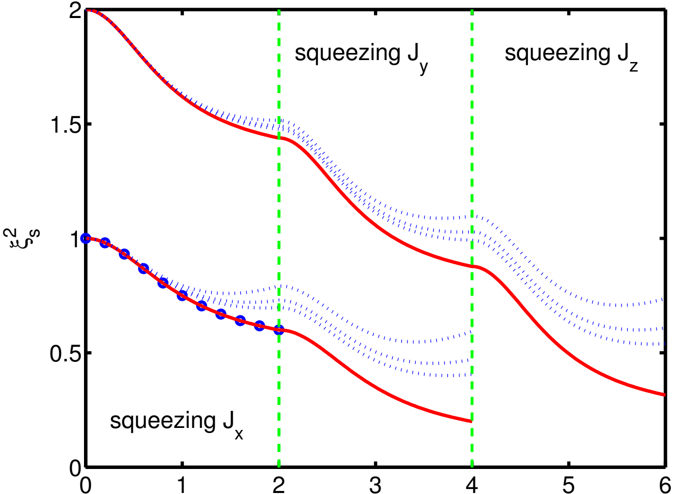
Experiments for creating many-body singlets with cold atomic ensembles



Picture from M.W. Mitchell, ICFO, Barcelona.

Experiments for creating many-body singlets

- Singlets can be created by squeezing the white noise
  [GT and M.W. Mitchell, NJP 2010.]
- We squeezed first J_x, then J_y, then J_z and obtain a singlet



Using singlets in metrology

- Singlet are invariant under the effect of homogenous magnetic fields.
- But are sensitive to field gradients.
- Thus, they can be used for differential magnetometry.
  [I. Urizar-Lanz, P. Hyllus, I. Egusquiza, GT, arxiv]
  [N. Behbood, B. Dubost, M. Napolitano, M. Koschorreck, R. Sewell, G. Tóth, and M.W. Mitchell, CEWQO 2011]
  [ongoing experiment in the Mitchell group at ICFO, Barcelona]

Symemtric Dicke states vs. singlets

- Comparing the basic properties of the two states we obtain the following.

	$\langle J_l \rangle$	$\langle J_x^2 \rangle + \langle J_y^2 \rangle$	$\langle J_z^2 \rangle$	H
$\vert\mathcal{D}_{N/2}^{(N)}\rangle$	0	max.	0	$-(J_x^2 + J_y^2)$
singlet	0	0	0	$(J_x^2 + J_y^2 + J_z^2)$

	$F_Q[\varrho, J_l]$	$\sum_{l=x,y,z} F_Q[\varrho, J_l]$
$\vert\mathcal{D}_{N/2}^{(N)}\rangle$	almost max.	max.
singlet	0	0

Group

Philipp Hyllus	Research Fellow
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Giuseppe Vitagliano	2nd year Ph.D. Student
Iagoba Appellaniz	M.Sc. Student (advisor P. Hyllus)

- Topics
  - Multipartite entanglement and its detection
  - Metrology, cold gases
  - Collaborating on experiments:
    - Weinfurter group, Munich, (photons)
    - Mitchell group, Barcelona, (cold gases)
- Funding:
  - European Research Council starting grant 2011-2016, 5-6 people for 5 years
  - CHIST-ERA QUASAR collaborative EU project
  - Grants of the Spanish Government and the Basque Government

Summary

- We discussed the interesting properties of Dicke states.
- We considered the symmetric Dicke state with $\langle J_z \rangle = 0$ is one of the most interesting states.
- We also considered singlet states.

THANK YOU FOR YOUR ATTENTION!

erc European Research Council

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