Entanglement and permutational symmetry

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Géza Tóth[1,2,3] and O. Gühne[4,5]

[1]Theoretical Physics, UPV/EHU, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Research Institute for Solid State Physics and Optics, Hungarian Academy of Sciences, Budapest

[4]Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften,Innsbruck, Austria
[5]Institut für Theoretische Physik, Universität Innsbruck, Innsbruck, Austria

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Outline

1. Motivation
2. Entanglement criteria for bipartite systems
3. Symmetric bound entangled states–Bipartite case
4. Symmetric bound entangled states–Multipartite case

Outline

Motivation

- Symmetry is a central concept in quantum mechanics. Typically, the presence of some symmetry simplifies our calculations in physics.
- We consider permutational symmetry.
- Can permutational symmetry simplify the separability problem?

Entanglement criteria for bipartite systems

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Two types of symmetries

Two d-dimensional quantum systems.

1. We call a state permutationally invariant (or just invariant, $\varrho \in \mathcal{I}$) if ϱ is invariant under exchanging the particles. That is,

$$F\varrho F = \varrho,$$

where the flip operator is $F = \sum_{ij} |ij\rangle\langle ji|$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry.

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2. We call a state symmetric ($\varrho \in \mathcal{S}$) if it acts on the symmetric subspace only.

$$F\varrho = \varrho F = \varrho.$$

This is the state space of two d-state bosons.

Clearly, we have $\mathcal{S} \subset \mathcal{I}$.

Connection to exchangeable states

- A state is exchangeable if it can be written as

$$\varrho = \int d\mu(\sigma)\sigma^{\otimes N}.$$

  Such a state has a symmetric extension of arbitrary number of qubits.

- Two-site marginals of such states are of the form

$$\sum_k p_k \varrho_k \otimes \varrho_k.$$

  They are permutationally invariant and separable states.

- There are permutionally invariant separable state that are not like that

$$\varrho = \frac{1}{2}(\varrho_1 \otimes \varrho_2 + \varrho_2 \otimes \varrho_1).$$

[M. Fannes and C. Vandenplas, J. Phys A 39, 13843 (2006).]

Expectation value matrix

Definition

Expectation value matrix of a bipartite quantum state is

$$\eta_{kl}(\varrho) := \langle M_k \otimes M_l \rangle_\varrho,$$

where M_k's are local orthogonal observables for both parties, satisfying

$$\mathrm{Tr}(M_k M_l) = \delta_{kl}.$$

- We can decompose the density matrix as

$$\varrho = \sum_{kl} \eta_{kl} M_k \otimes M_l.$$

Equivalence of several entanglement conditions for symmetric states

Observation 1. Let $\varrho \in \mathcal{S}$ be a symmetric state. Then the following separability criteria are equivalent:

1. ϱ has a positive partial transpose (PPT), $\varrho^{T_A} \geq 0$.
2. ϱ satisfies the Computable Cross Norm-Realignment (CCNR) criterion, $||R(\varrho)||_1 \leq 1$, where $R(\varrho)$ is the realignment map and $||...||_1$ is the trace norm.
3. $\eta \geq 0$, or, equivalently $\langle A \otimes A \rangle \geq 0$ for all observables A.
4. The correlation matrix, defined via the matrix elements as

   $$C_{kl} := \langle M_k \otimes M_l \rangle - \langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle$$

   is positive semidefinite: $C \geq 0$. [A.R. Usha Devi *et al.*, Phys. Rev. Lett. **98**, 060501 (2007).]
5. The state satisfies several variants of the Covariance Matrix Criterion (CMC). Latter are strictly stronger than the CCNR criterion for non-symmetric states.

Proof of Observation 1: Schmidt decomposition

Proof.

- For invariant states, η is a real symmetric matrix.
- Diagonalization: $\{\Lambda_k\}$ is the correlation matrix corresponding to the observables $M'_k = \sum O_{kl} M_l$.

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$$\varrho = \sum_k \Lambda_k M'_k \otimes M'_k,$$

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  where M'_k are pairwise orthogonal observables. This is almost the Schmidt decomposition, however, Λ_k can also be negative.
- It can be shown that $-1 \leq \sum_k \Lambda_k \leq 1$ for invariant states and $\sum_k \Lambda_k = 1$ for symmetric states.

Proof of Observation 1: Equivalence of CCNR and $\eta \geq 0$

Now we can show the first equivalences.

- The Computable Cross Norm-Realignment (CCNR) can be formulated as follows: If

$$\sum_k |\Lambda_k| > 1$$

in the Schmidt decomposition, then the quantum state is entangled.

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- For symmetric states we have $\sum_k \Lambda_k = 1$, and $\sum_k |\Lambda_k| > 1$ is equivalent to

$$\Lambda_k < 0$$

  for some k. Then $\langle M'_k \otimes M'_k \rangle < 0$ and η has a negative eigenvalue.

Proof of Observation 1: CCNR–PPT equivalence

Let us take an alternative definition of the CCNR criterion.

- The CCNR criterion states that if ϱ is separable, then $\|R(\varrho)\|_1 \leq 1$ where the realigned density matrix is $R(\varrho_{ij,kl}) = \varrho_{ik,jl}$. This just means that if

$$\|(\varrho F)^{T_A}\|_1 > 1$$

then ϱ is entangled.

[M.M. Wolf, Ph.D. Thesis, TU Braunschweig, 2003.]

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- Since for symmetric states

  $$\varrho F = \varrho,$$

  this condition is equivalent to $\|\varrho^{T_A}\|_1 > 1$. This is just the PPT criterion, since we have $\mathrm{Tr}(\varrho^{T_A}) = 1$.

Proof of Observation 1: Equivalence of $C \geq 0$ and $\eta \geq 0$

- Now we show that $C \geq 0 \Leftrightarrow \eta \geq 0$.
- The direction "$\Rightarrow$" is trivial, since for invariant states the matrix $\langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle$ is a projector and hence positive.
- The direction "$\Leftarrow$": We make for a given state the special choice of observables $Q_k = M_k - \langle M_k \rangle$. Then, we just have $C(M_k) = \eta(Q_k)$.
- We know that $\eta(M_k) \geq 0 \Rightarrow \eta(Q_k) \geq 0$, even if Q_k are not pairwise orthogonal observables. Hence $C(M_k) \geq 0$ follows.

Proof of Observation 1: Covariance Matrix Criterion

- Variants of the Covariance Matrix Criterion:

$$\|C\|_1^2 \leq [1 - \mathrm{Tr}(\varrho_A^2)][1 - \mathrm{Tr}(\varrho_B^2)]$$

or

$$2\sum |C_{ii}| \leq [1 - \mathrm{Tr}(\varrho_A^2)] + [1 - \mathrm{Tr}(\varrho_B^2)].$$

[O. Gühne *et al.*, PRL **99**, 130504 (2007); O. Gittsovich *et al.*, PRA **78**, 052319 (2008).]

- If ϱ is symmetric, the fact that C is positive semidefinite gives $\|C\|_1 = \mathrm{Tr}(C) = \sum \Lambda_k - \sum_k \mathrm{Tr}(\varrho_A M_k')^2 = 1 - \mathrm{Tr}(\varrho_A^2)$ and similarly, $\sum_i |C_{ii}| = \sum_i C_{ii} = 1 - \mathrm{Tr}(\varrho_A^2)$.

- Hence, a state fulfilling $C \geq 0$ fulfills also both criteria. On the other hand, a state violating $C \geq 0$ must also violate these criteria, as they are strictly stronger than the CCNR criterion

Consequences

- Interesting result: For symmetric ϱ
$$\varrho^{T1} \geq 0 \iff \forall A : \langle A \otimes A \rangle \geq 0.$$
This relates the positivity of partial transposition to the sign of certain two-body correlations.

- Any symmetric state of the following form is PPT
$$\varrho_{\mathrm{PPT}} = \sum_k p_k M_k \otimes M_k, \tag{1}$$
where p_k is a probability distribution, and M_k are pairwise orthogonal observables, i.e., $\mathrm{Tr}(M_k^2) = 1$. Compare this to the definition of separability
$$\varrho_{\mathrm{sep}} = \sum_k p_k \varrho_k \otimes \varrho_k, \tag{2}$$
where ϱ_k are observables, $\mathrm{Tr}(\varrho_k) = 1$, $\varrho_k \geq 0$ and ϱ_k are pure, i.e, $\mathrm{Tr}(\varrho_k^2) = 1$.

Consequences II

- Any symmetric state that can be written as

$$\varrho_{c+} = \sum_k c_k A_k \otimes Ak, \tag{3}$$

where $c_k > 0$, and A_k are some (not necessarily pairwise orthogonal) observables, is PPT. If ϱ_{c+} is permutationally invariant, then it does not violate the CCNR criterion.

- Multipartite case: A symmetric state of the form

$$\varrho_{\mathrm{PPT2:2}} = \sum_k c_k A_k \otimes A_k \otimes A_k \otimes A_k \tag{4}$$

is PPT with respect to the 2 : 2 partition. Example: Smolin state.

Consequences III

- Relation to separability. Symmetric separable states:

$$\varrho_{\rm sep} = \sum_k p_k \varrho_k \otimes \varrho_k.$$

For such states,

$$\mathrm{Tr}(A \otimes A \varrho_{\rm sep}) = \sum_k p_k \mathrm{Tr}(A \varrho_k)^2 \geq 0.$$

Thus

$$\varrho \text{ is separable} \Rightarrow \forall A : \langle A \otimes A \rangle_\varrho \geq 0.$$

But not " $\Longleftrightarrow$ " !

Consequences IV

- Relation to decomposability. Permutationally invariant matrix:

$$M = \sum_k c_k M_k \otimes M_k.$$

For such matrices

$$\exists\{c_k \geq 0\} : M = \sum_k c_k M_k \otimes M_k \iff \forall A : \langle A \otimes A \rangle_\varrho \geq 0.$$

Now we have " $\Longleftarrow$ " !

Are there symmetric bound entangled states?

- For symmetric states,
  1. CCNR,
  2. $\eta \geq 0$,
  3. $C \geq 0$ and
  4. CMC

  are equivalent to the PPT criterion.
- It is then quite hard to find symmetric PPT entangled states.

Do symmetric bound entangled states exist at all?


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Symmetric bound entangled states–Bipartite case

Symmetric bound entangled states

- Breuer presented, for even $d \geq 4$, a single parameter family of bound entangled states that are $\mathcal{I}$ symmetric

$$\varrho_{\mathrm{B}} = \lambda|\Psi_0^d\rangle\langle\Psi_0^d| + (1-\lambda)\Pi_s^d.$$

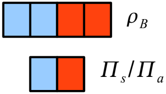
[H.-P. Breuer, PRL **97**, 080501 (2006); see also K.G.H. Vollbrecht and M.M. Wolf, PRL **88**, 247901 (2002).]

- The state is PPT entangled for $0 \leq \lambda \leq 1/(d+2)$. Here $|\Psi_0\rangle$ is the singlet state and Π_s is the normalized projector to the symmetric subspace.

- Idea to construct bound entangled states with an $\mathcal{S}$-symmetry: We embed a low dimensional entangled state into a higher dimensional Hilbert space, such that it becomes symmetric, while it remains entangled.

An 8×8 symmetric bound entangled states

- We consider the symmetric state



$$\hat{\varrho} = \lambda \Pi_a^{d_2} \otimes |\Psi_0^d\rangle\langle\Psi_0^d| + (1-\lambda)\Pi_s^{d_2} \otimes \Pi_s^d.$$

Here, $\Pi_a^{d_2}$ and $\Pi_s^{d_2}$ are normalized projectors to the two-qudit symmetric/antisymmetric subspace with dimension d_2. Thus, $\hat{\varrho}$ is symmetric.

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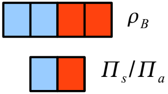

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- If the original system is of dimension $d \times d$ then the system of $\hat{\varrho}$ is of dimension $dd_2 \times dd_2$. Since ϱ_{B} is the reduced state of $\hat{\varrho}$, if the first is entangled, then the second is also entangled.

- For $d_2 = 2$ and $d = 4$, numerical calculation shows that $\hat{\varrho}$ is PPT for $\lambda < 0.062$.

This provides an example of an $\mathcal{S}$ symmetric bound entangled state of size 8×8.

Symmetric bound entangled state via numerics– Basic idea

- An N-qubit symmetric state, that is, a state of the symmetric subspace (even N).

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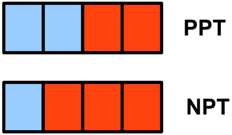
- Thus any state that is PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition while NPT with respect to some other partition is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition.



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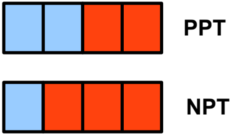

- Since the state is symmetric, it can straightforwardly be mapped to a $(\frac{N}{2}+1)\times(\frac{N}{2}+1)$ bipartite symmetric state.

Symmetric bound entangled state via numerics II

- Fist, we generate an initial random state ϱ that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition.

Symmetric bound entangled state via numerics II

- Fist, we generate an initial random state ϱ that is PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition.
- Then, we compute the minimum nonzero eigenvalue of the partial transpose of ϱ with respect to all other partitions

$$\lambda_{\min}(\varrho) := \min_k \min_l \lambda_l(\varrho^{T_{I_k}}).$$

If $\lambda_{\min}(\varrho) < 0$ then the state is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition. If it is non-negative then we start an optimization process for decreasing this quantity.

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- We generate another random density matrix $\Delta\varrho$, and check the properties of

$$\varrho' = (1 - \varepsilon)\varrho + \varepsilon\Delta\varrho, \tag{5}$$

  where $0 < \varepsilon < 1$ is a small constant. If ϱ' is still PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition and $\lambda_{\min}(\varrho') < \lambda_{\min}(\varrho)$ then we use ϱ' as our new random initial state ϱ.

3×3 symmetric bound entangled state

- Repeating this procedure, we obtained a four-qubit symmetric state this way

$$\varrho_{BE4} = \begin{pmatrix} 0.22 & 0 & 0 & -0.059 & 0 \\ 0 & 0.176 & 0 & 0 & 0 \\ 0 & 0 & 0.167 & 0 & 0 \\ -0.059 & 0 & 0 & 0.254 & 0 \\ 0 & 0 & 0 & 0 & 0.183 \end{pmatrix}.$$

  The basis states are $|0\rangle := |0000\rangle$, $|1\rangle := \mathrm{sym}(|1000\rangle)$, $|2\rangle := \mathrm{sym}(|1100\rangle), \ldots$

- The state is bound entangled with respect to the $2:2$ partition. This corresponds to a 3×3 bipartite symmetric bound entangled state.

- Simplest possible symmetric bound entangled state

Five- and six-qubit fully PPT entangled states

- Our method can be straightforwardly generalized to create multipartite bound entangled states of the symmetric subspace, such that *all* bipartitions are PPT (“fully PPT states”).
- We found such a state for five and six qubits.
- These states are both fully PPT and genuine multipartite entangled.
- Peres conjecture: fully PPT states cannot violate a Bell inequality.


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Conclusions

- In summary, we have discussed entanglement in symmetric systems.
- We showed that for states that are in the symmetric subspace several relevant entanglement condition coincide:
  - PPT criterion
  - CCNR criterion
  - $\eta \geq 0$
  - $C \geq 0$
  - CMC
- We proved the existence of symmetric bound entangled states, in particular, 3×3, five-qubit and six-qubit symmetric PPT entangled states.
- See G. Tóth and O. Gühne, PRL 102, 170503 (2009).

*** THANK YOU ***