Spin squeezing and entanglement

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Outline

1. Motivation
2. Entanglement detection with collective observables
3. Optimal spin squeezing inequalities
4. Multipartite bound entanglement in spin models

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Motivation

- In many quantum control experiments the qubits cannot be individually accessed. We still would like to detect entanglement.
- The spin squeezing criterion is already known. Are there other similar criteria that detect entanglement with the first and second moments of collective observables?
- Generalized spin squeezing inequalities might help identifying the entangled states useful for metrology.

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Spin squeezing

1. Spin squeezing, according to the original definition, is interpreted in the following context. The variances of the angular momentum components are bounded by the following uncertainty relation

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{|\langle Jz\rangle|}{2}$ then the state is called spin squeezed.

2. In practice this means that the angular momentum of the state has a small variance in one direction, while in an orthogonal direction the angular momentum is large.

[M. Kitagawa and M. Ueda, PRA **47**, 5138 (1993).]

Definition of entanglement

- Fully separable states are states that can be written in the form
$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes ... \otimes \rho_l^{(N)},$$
where $\sum_l p_l = 1$ and $p_l > 0$.

- A state is entangled if it is not separable.

- Note that one could also look for other type of entanglement in many-particle systems, e.g., entanglement in the two-qubit reduced density matrix.

Collective quantities

- What if we cannot address the particles individually? This is expected to occur often in future experiments.
- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices. We can also measure the $(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$ variances.

The standard spin-squeezing criterion

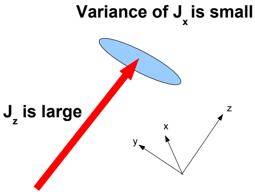
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature **409**, 63 (2001).]

- States violating it are like this:



Generalized spin squeezing entanglement criteria I

- Separable states must fulfill

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}.$$

  It is maximally violated by a many-body singlet, e.g., the ground state of an anti-ferromagnetic Heisenberg chain.

  [GT, PRA **69**, 052327 (2004).]

- For such a state

$$\langle J_k^m \rangle = 0.$$

- Note that there are very many states giving zero for the left hand side. The mixture of all such states also maximally violates the criterion.

Generalized spin squeezing entanglement criteria II

- For states with a separable two-qubit density matrix

$$\left(\langle J_k^2\rangle + \langle J_l^2\rangle - \frac{N}{2}\right)^2 + (N-1)^2\langle J_m\rangle^2 \leq \langle J_m^2\rangle + \frac{N(N-2)}{4}$$

holds.

[J. Korbicz, I. Cirac, M. Lewenstein, PRL **95**, 120502 (2005).]

- Detects all symmetric two-qubit entangled states; can be used to detect symmetric Dicke states.

- Used in ion trap experiment.

[J. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C.F. Roos, R. Blatt, PRA 74, 052319 (2005).]

Generalized spin squeezing entanglement criteria III

- For separable states [GT, J. Opt. Soc. Am. B **24**, 275 (2007).]

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq \frac{N(N+1)}{4}$$

  holds.

- This can be used to detect entanglement close to N-qubit symmetric Dicke states with $\frac{N}{2}$ excitations. For such a state

$$\begin{aligned} \langle J_k\rangle &= 0, \\ \langle J_z^2\rangle &= 0, \\ \langle J_{x/y}^2\rangle &= \frac{N(N+2)}{8}. \end{aligned}$$

- For $N = 4$, this state looks like

$$|\Psi\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle).$$

  This was realized with photons.

  [N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 98, 063604 (2007).]

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Optimal spin squeezing inequalities

- Let us assume that for a system we know only

$$\mathbf{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\mathbf{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \tfrac{N(N+2)}{4}, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \tfrac{N}{2}, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \tfrac{N}{2}, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + \tfrac{N(N-2)}{4},
\end{aligned}$$

where k, l, m take all the possible permutations of x, y, z.

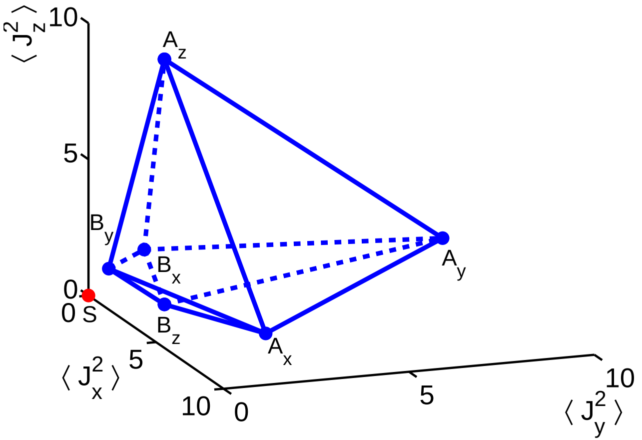
[GT, C. Knapp, O. Gühne, és H.J. Briegel, PRL 99, 250405 (2007); quant-ph/0702219.]

Derivation of the equations

- Criterion 2
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
Proof: For product states
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = \sum_k (\Delta j_x^{(k)})^2 + (\Delta j_y^{(k)})^2 + (\Delta j_z^{(k)})^2 \geq \frac{N}{2}.$$
It is also true for separable states due to the convexity of separable states.
- Criterion 3
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
Proof: For product states
$$(N-1)(\Delta J_x)^2 + \frac{N}{2} - \langle J_y^2 \rangle - \langle J_z^2 \rangle = (N-1)\left(\frac{N}{4} - \frac{1}{4}\sum_k x_k^2\right)$$
$$-\frac{1}{4}\sum_{k \neq l} y_k y_l + z_k z_l = ... \geq 0.$$
Here $x_k = \langle \sigma_x^{(k)} \rangle$ and we have to use $(\sum_k s_k)^2 \leq N \sum_k s_k$.

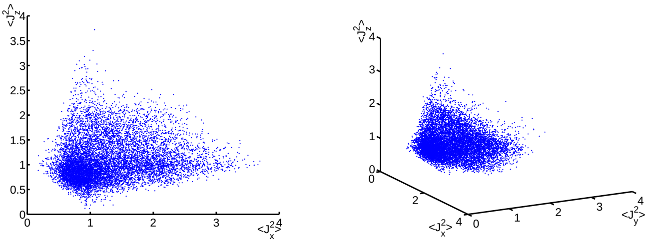
The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space. The polytope has six extreme points: $A_{x/y/z}$ and $B_{x/y/z}$.
- For $\langle \mathbf{J} \rangle = 0$ and $N = 6$ the polytope is the following:



The polytope II

- Random separable states:



The polytope III

- The coordinates of the extreme points are

$$A_x := \left[\frac{N^2}{4} - \kappa(\langle J_y\rangle^2 + \langle J_z\rangle^2), \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

$$B_x := \left[\langle J_x\rangle^2 + \frac{\langle J_y\rangle^2 + \langle J_z\rangle^2}{N}, \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

  where $\kappa := (N-1)/N$. The points $A_{y/z}$ and $B_{y/z}$ can be obtained from these by permuting the coordinates.

- Now it is easy to prove that an inequality is a necessary condition for separability: All the six points must satisfy it.

The polytope IV

- Let us take the $\langle \mathbf{J} \rangle = 0$ case first.
- Then the state corresponding to A_x is the equal mixture of

$$|+1,+1,+1,+1,...\rangle_x$$

and

$$|-1,-1,-1,-1,...\rangle_x.$$

- The state corresponding to B_x is

$$|+1\rangle_x^{\otimes \frac{N}{2}} \otimes |-1\rangle_x^{\otimes \frac{N}{2}}.$$

- Separable states corresponding to $A_{y/z}$ and $B_{y/z}$ are defined similarly.

The polytope V

- General case: $\langle \mathbf{J} \rangle \neq 0$.
- A separable state corresponding to A_x is

$$\rho_{A_x} = p(|\psi_+\rangle\langle\psi_+|)^{\otimes N} + (1-p)(|\psi_-\rangle\langle\psi_-|)^{\otimes N}.$$

  Here $|\psi_{+/-}\rangle$ are the single qubit states with Bloch vector coordinates $(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = (\pm c_x, 2\langle J_y\rangle/N, 2\langle J_z\rangle/N)$ where $c_x := \sqrt{1 - 4(\langle J_y\rangle^2 + \langle J_z\rangle^2)/N^2}$. The mixing ratio is defined as $p := 1/2 + \langle J_x\rangle/(Nc_x)$.
- If $N_1 := Np$ is an integer, we can also define the state corresponding to the point B_x as

$$|\phi_{B_x}\rangle = |\psi_+\rangle^{\otimes N_1} \otimes |\psi_-\rangle^{\otimes(N-N_1)}.$$

  If N_1 is not an integer then one can find a point B_x' such that such that its distance from B_x is smaller than $\frac{1}{4}$.

In what sense is the characterization complete?

- For any value of **J** there are separable states corresponding to $A_{x/y/z}$.
- For certain values of **J** and N (e.g., $\mathbf{J} = 0$ and even N) there are separable states corresponding to points $B_{x/y/z}$.
- However, there are always separable states corresponding to points $B'_{x/y/z}$ such that their distance from $B_{x/y/z}$ is smaller than $\frac{1}{4}$.
- In the limit $N \to \infty$ for a fixed normalized angular momentum $\frac{\mathbf{J}}{N/2}$ the difference between the volume of our polytope and the volume of set of points corresponding to separable states decreases with N as

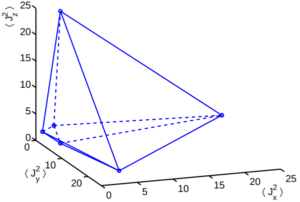
  $$\frac{\Delta V}{V} \propto N^{-2},$$

  hence in the macroscopic limit the characterization is complete.

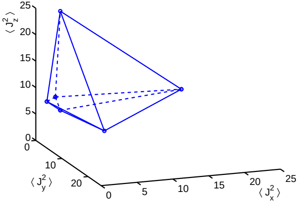
Polytope for various values for J

- The polytope for $N = 10$ and

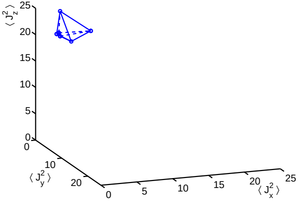
$J = (0, 0, 0)$,



$J = (0, 0, 2.5)$,



and $J = (0, 0, 4.5)$.

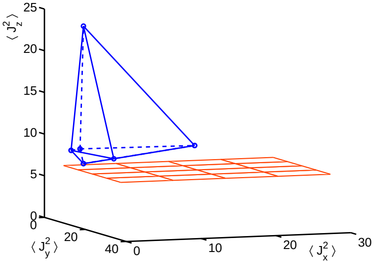


Our inequalities vs. the standard spin squeezing criterion

- The standard spin squeezing criterion

$$\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2} \geq \frac{1}{N}$$

is satisfied by all points A_k and B_k, for B_z even equality holds.



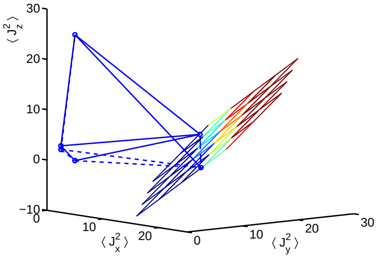
- Polytope for $N = 10$ and $J = (1.5, 0, 2.5)$. States that are detected by the standard criterion are below the red plane.

Our inequalities vs. the Korbicz-Cirac-Lewenstein inequalities

- For states with a separable two-qubit density matrix

$$\left(\langle J_k^2\rangle + \langle J_l^2\rangle - \frac{N}{2}\right)^2 + (N-1)^2\langle J_m\rangle^2 \leq \langle J_m^2\rangle + \frac{N(N-2)}{4}$$

holds. [J. Korbicz *et al.* PRL **95**, 120502 (2005).]



- Polytope for $N = 10$ and $J = (0, 0, 0)$. States that are detected by the KCL criterion are below the plane. The plane contains two of the three A_k points.

Correlation matrix

- Our inequalities can be reexpressed with the correlation matrix.
- Basic definitions:

$$\begin{aligned} C_{kl} &:= \tfrac{1}{2}\langle J_k J_l + J_l J_k \rangle, \\ \gamma_{kl} &:= C_{kl} - \langle J_k \rangle \langle J_l \rangle. \end{aligned} \tag{2}$$

- With them we define the interesting quantity

$$\mathfrak{X} := (N-1)\gamma + C. \tag{3}$$

Correlation matrix II

- Now we can rewrite our inequalities as

$$
\begin{aligned}
\mathrm{Tr}(\mathfrak{X}) &\leq \tfrac{N^2(N+2)}{4} - (N-1)|\mathbf{J}|^2, \\
\mathrm{Tr}(\mathfrak{X}) &\geq \tfrac{N^2}{2} + |\mathbf{J}|^2, \\
\lambda_{\min}(\mathfrak{X}) &\geq \tfrac{1}{N}\mathrm{Tr}(\mathfrak{X}) + \tfrac{N-1}{N}|\mathbf{J}|^2 - \tfrac{N}{2}, \\
\lambda_{\max}(\mathfrak{X}) &\leq \tfrac{N-1}{N}\mathrm{Tr}(\mathfrak{X}) - \tfrac{N-1}{N}|\mathbf{J}|^2 - \tfrac{N(N-2)}{4},
\end{aligned}
$$

  For fixed $|\mathbf{J}|$ these equations describe a polytope in the space of the three eigenvalues of $\mathfrak{X}$.
- These new inequalities detect all entangled quantum states that can be detected based on knowing the correlation matrix and $\mathbf{J}$.

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Two-qubit entanglement

- Our criteria can detect entangled states for which the reduced two-qubit density matrix is separable.

- This might look surprising since all our criteria contain operator expectation values that can be computed knowing the average two-qubit density matrix

$$\rho_{12} := \frac{1}{N(N-1)} \sum_{k \neq l} \rho_{kl},$$

  and no information on higher order correlation is used.

- Still, our criteria do not merely detect entanglement in the reduced two-qubit state!

Two-qubit entanglement II

- Two-qubit symmetric separable states have the form

$$\rho_{12} = \sum_k p_k \rho_k \otimes \rho_k.$$

  For such states it is always possible to find an N-qubit separable state, which has ρ_{12} as it reduced state:

$$\rho = \sum_k p_k \rho_k \otimes \rho_k \otimes ... \otimes \rho_k.$$

  Note the connection to the representability problem.

- However, there are two-qubit separable states for which this is not possible. For example, these can be of the form

$$\rho_{12} = \frac{1}{2}(\rho_1 \otimes \rho_2 + \rho_2 \otimes \rho_1).$$

  Clearly, it is not easy to find an N-qubit state for such a state.

Two-qubit entanglement III

- From the previous discussion it follows the following:

  For symmetric states, the violation of any entanglement criterion with $\langle J_k \rangle$ and $\langle J_k^2 \rangle$ implies the entanglement of the reduced two-qubit density matrix.

- This was found by Wang and Sanders for the standard spin-squeezing inequality.

  [X. Wang and B.C. Sanders, PRA **68**, 012101 (2003).]

Bound entanglement in spin chains

- Let us consider four spin-1/2 particles, interacting via the Hamiltonian

$$H = (h_{12} + h_{23} + h_{34} + h_{41}) + J_2(h_{13} + h_{24}),$$

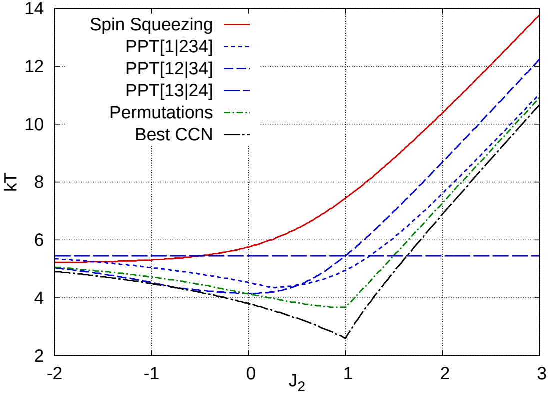
  where $h_{ij} = \sigma_x^{(i)} \otimes \sigma_x^{(j)} + \sigma_y^{(i)} \otimes \sigma_y^{(j)} + \sigma_z^{(i)} \otimes \sigma_z^{(j)}$ is a Heisenberg interaction between the qubits i, j.

- For the above Hamiltonian we compute the thermal state $\varrho(T, J_2) \propto \exp(-H/kT)$ and investigate its separability properties.

- For several separability criteria we calculate the maximal temperature, below which the criteria detect the states as entangled.

Bound entanglement in spin systems II

- Bound temperatures for entanglement



For $J_2 \gtrsim -0.5$, the spin squeezing inequality is the strongest criterion for separability. It allows to detect entanglement even if the state has a positive partial transpose (PPT) with respect to all bipartition.

Bound entanglement in spin systems III

- We found bound entanglement that is PPT with respect to all bipartitions in XY and Heisenberg chains, and also in XY and Heisenberg models on a completely connected graph, up to 10 qubits.
- Thus for these models, which appear in nature, there is a considerable temperature range in which the system is already PPT but not yet separable.

Bound entanglement in spin systems IV

- Simple example: Heisenberg system on a fully connected graph

$$H = J_x^2 + J_y^2 + J_z^2 = \frac{3N}{4} + \frac{1}{4}\sum_{k\neq l} \sigma_x^{(k)}\sigma_x^{(l)} + \sigma_y^{(k)}\sigma_y^{(l)} + \sigma_z^{(k)}\sigma_z^{(l)}.$$

- The ground state is very mixed: For large temperature range it is PPT bound entangled.
- The thermodynamics of this system can be computed analytically. Optimal spin squeezing inequalities are violated for $T < N$. [GT, PRA 71, 010301(R) (2005).]

Conclusions

- We presented a family of entanglement criteria that are able to detect any entangled state that can be detected based on the first and second moments of collective angular momenta.
- We explicitly determined the set of points corresponding to separable states in the space of first and second order moments.
- We applied our findings to examples of spin models, showing the presence of bound entanglement in these models.

*** THANK YOU ***