Spin-squeezing inequalities for entanglement detection in cold gases Phys. Rev. Lett. 107, 240502 (2011)

G. Tóth[1,2,3], G. Vitagliano[1], P. Hyllus[1], and I.L. Egusquiza[1],

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary

ICSSUR 2013, Nürnberg,
24 June 2013


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Barents Sea
Hammerfest
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Norwegian Sea
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Oran
Algiers
Tunis
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
0 300 Kilometers
0 20 300 Miles

Outline

1. **Motivation**
   - Why spin squeezing inequalities are important?
2. **Multipartite entanglement**
   - Definition of entanglement
3. **Spin squeezing entanglement criteria for $j = 1/2$**
   - Collective measurements
   - The original criterion
   - Generalized criteria for $j = \frac{1}{2}$
4. **Spin squeezing inequality for an ensemble of spin-j atoms**
   - Conditions with the angular momentum coordinates for $j > \frac{1}{2}$
   - The usual spin squeezing inequality for $j > \frac{1}{2}$
   - Conditions with the SU(d) generators
   - Detection of singlets

Why spin squeezing inequalities for $j > \frac{1}{2}$ is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Most experiments use atoms with $j > \frac{1}{2}$.

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Entanglement

Definition

A multiparticle state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not fully separable, then it is called entangled.

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Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \tfrac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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The standard spin-squeezing criterion

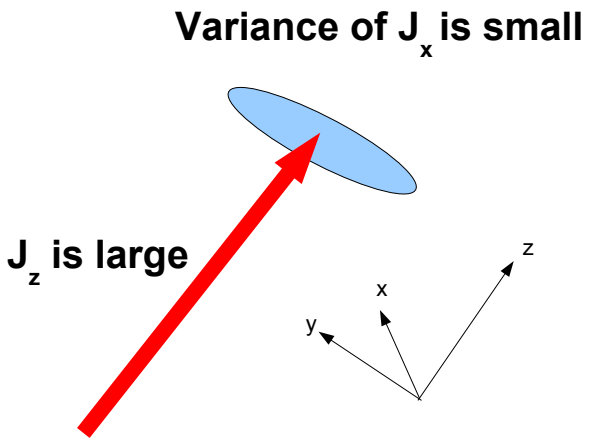
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

- If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States violating it are like this:



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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled.

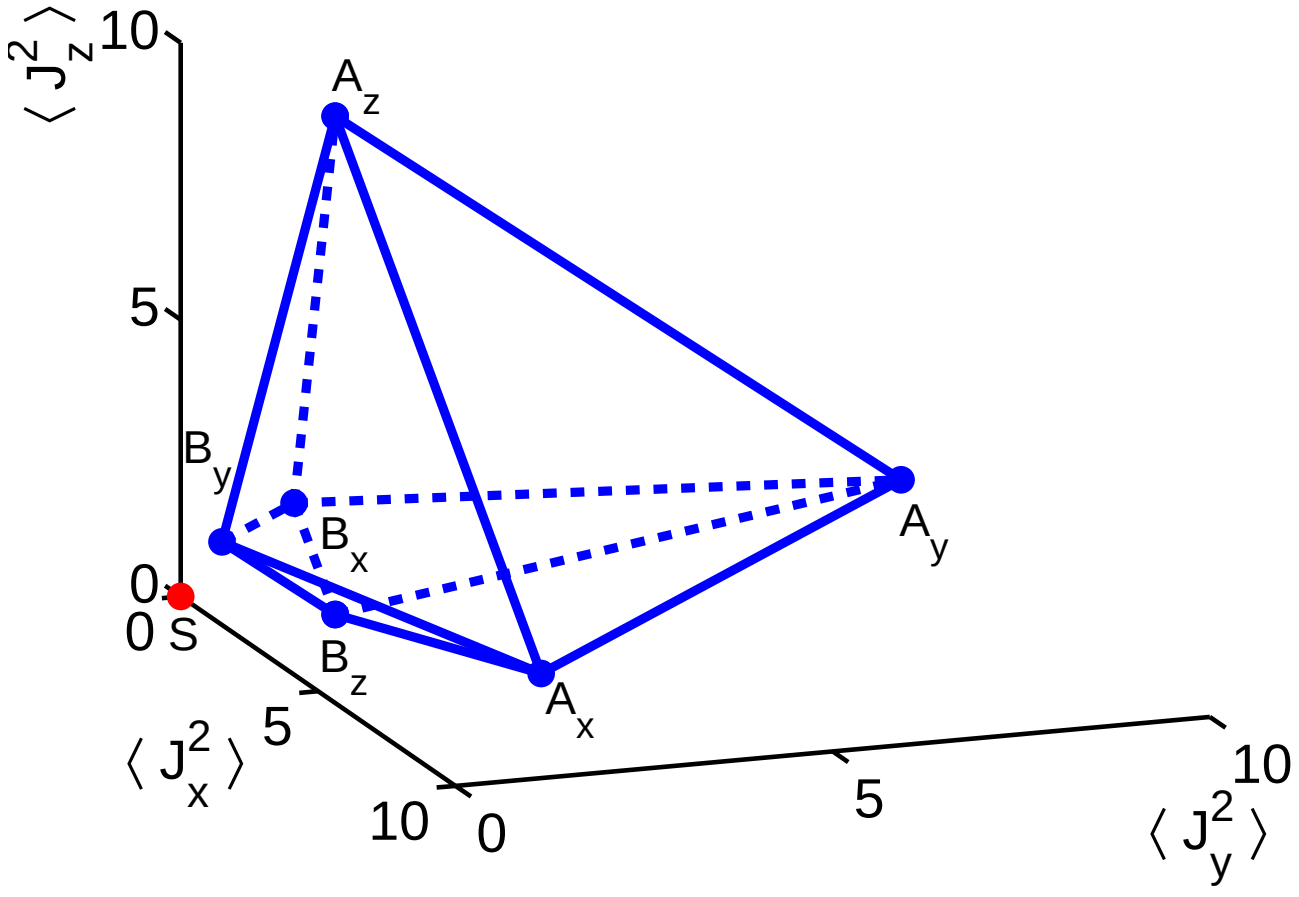
$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4}, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2}, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},
\end{aligned}$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

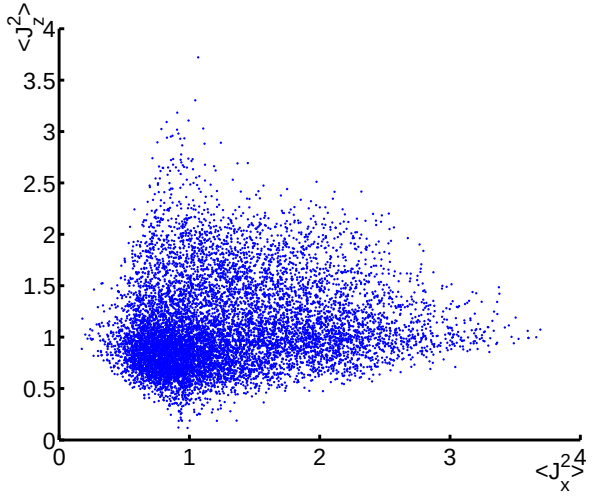
Generalized spin squeezing criteria for $j = \frac{1}{2}$

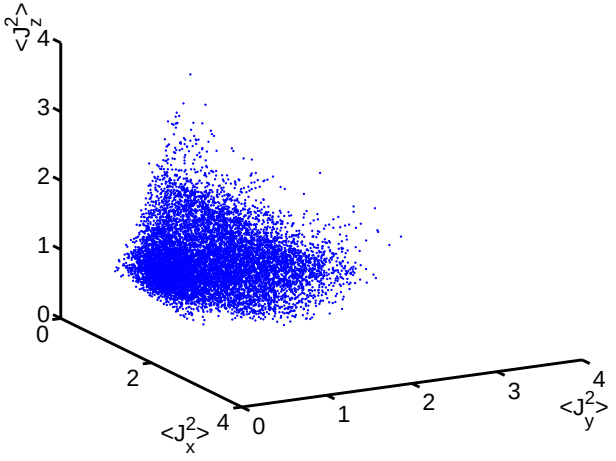
- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- For $\left\langle \vec{J} \right\rangle = 0$ and $N = 6$ the polytope is the following:



Completeness

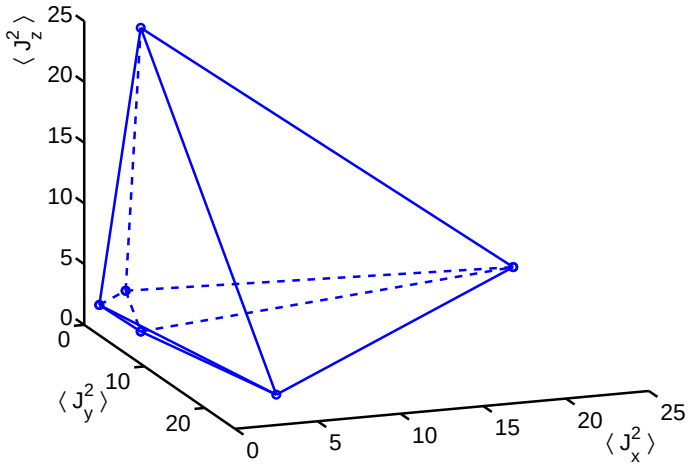
- Random separable states:



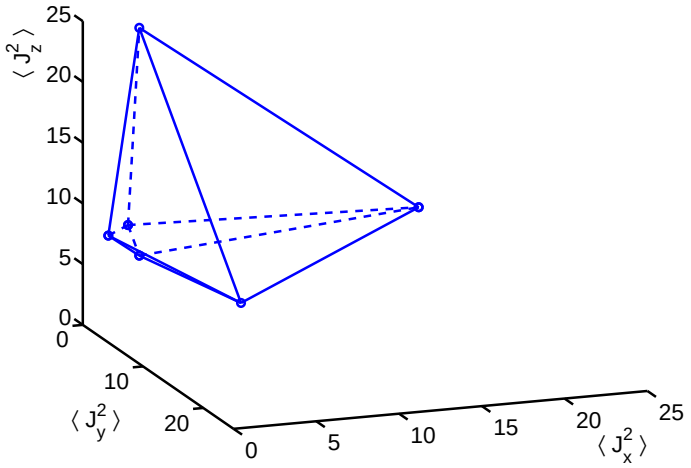


- The completeness can be proved for large *N*.

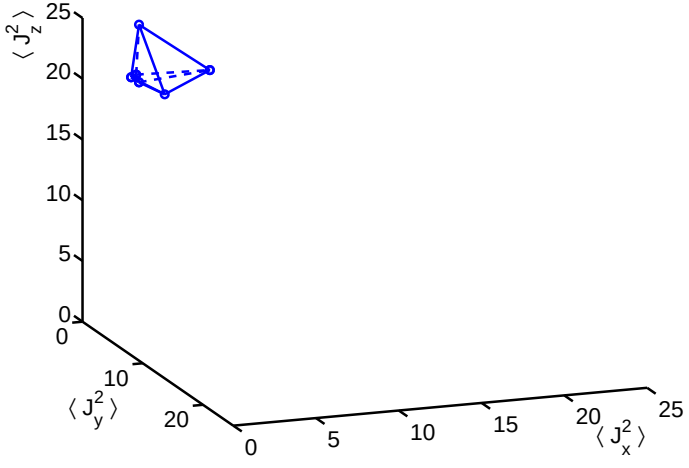
- The polytope for $N = 10$ and $J = (0, 0, 0)$,



$J = (0, 0, 2.5)$,



and $J = (0, 0, 4.5)$.



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"Modified" quantities for $j > \frac{1}{2}$

- For the $j = \frac{1}{2}$ case, the SSIs were developed based on the first and second moments and variances of the such collective operators.
- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{J}_k^2 \rangle := \langle J_k^2 \rangle - \langle \sum_n (j_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle j_k^{(n)} j_k^{(m)} \rangle$$

  and the modified variance

$$(\tilde{\Delta} J_k)^2 := (\Delta J_k)^2 - \langle \sum_n (j_k^{(n)})^2 \rangle.$$

- These are essential to get tight equations for $j > \frac{1}{2}$.

The inequalities for $j>\frac{1}{2}$ with the angular momentum coordinates

- For fully separable states of spin-j particles, all the following inequalities are fulfilled

$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\leq Nj(Nj+1), \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj, \\
\langle \tilde{J}_k^2\rangle + \langle \tilde{J}_l^2\rangle - N(N-1)j^2 &\leq (N-1)(\tilde{\Delta} J_m)^2, \\
(N-1)\left[(\tilde{\Delta} J_k)^2 + (\tilde{\Delta} J_l)^2\right] &\geq \langle \tilde{J}_m^2\rangle - N(N-1)j^2,
\end{aligned}$$

  where k, l, m take all possible permutations of x, y, z.

- Violation of any of the inequalities implies entanglement.

Completeness

- In the large N limit, the inequalities with the angular momentum are complete.
- It is not possible to find new entanglement conditions based on $\langle J_k \rangle$ and $\langle \tilde{J}_k^2 \rangle$ that detect more states.

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The usual spin squeezing inequality for $j > \frac{1}{2}$

- The standard spin-squeezing inequality becomes

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} + \frac{\sum_n (j^2 - \langle (j_x^{(n)})^2\rangle)}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

  Violated only if there is entanglement between the spin-j particles.

- The second term on the LHS is nonnegative.

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The inequalities for $j > \frac{1}{2}$ with the G_k's

- Collective operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

where $l = 1, 2, ..., d^2 - 1$ and $g_l^{(k)}$ are the SU(d) generators.

- We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.

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An example: The criterion for SU(d) singlets

A condition for two-producibility (i.e., higher form of entanglement) for N qudits of dimension d is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Spin squeezing inequalities for arbitrary spin, PRL 2011.]

Group

Philipp Hyllus	Research Fellow (2011-2012)
Zoltán Zimborás	Research Fellow (2012-)
Iñigo Urizar-Lanz	Ph.D. Student
Giuseppe Vitagliano	Ph.D. Student
Iagoba Apellaniz	Ph.D. Student

- Topics
  - Multipartite entanglement and its detection
  - Metrology, cold gases
  - Collaborating on experiments:
    - Weinfurter group, Munich, (photons)
    - Mitchell group, Barcelona, (cold gases)
- Funding:
  - European Research Council starting grant 2011-2016, 1.3 million euros
  - CHIST-ERA QUASAR collaborative EU project (H. Weinfurter)
  - Grants of the Spanish Government and the Basque Government

Summary

- Full set of generalized spin squeezing inequalities with J_l with $l = x, y, z$ for $j > \frac{1}{2}$.
- Large set of inequalities with the other collective operators.
- These might make possible new experiments and make existing experiments simpler.

See: G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Phys. Rev. Lett. 107, 240502 (2011) + manuscript in preparation.

See `www.gtoth.eu`

THANK YOU FOR YOUR ATTENTION!

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