Entanglement and permutational symmetry
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Outline

1. Motivation
2. Entanglement criteria for bipartite systems
3. Symmetric bound entangled states–Bipartite case
4. Symmetric bound entangled states–Multipartite case

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Motivation

- Symmetry is a central concept in quantum mechanics. Typically, the presence of some symmetry simplifies our calculations in physics.
- A particular type of symmetry, permutational symmetry appears in many systems studied in quantum optics.
- The separability problem is proven to be a very hard one. Thus, it is interesting to ask how permutational symmetry can simplify the separability problem.

Entanglement criteria for bipartite systems

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3. Symmetric bound entangled states–Bipartite case
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Two types of symmetries

Consider two d-dimensional quantum systems. We will examine two types of permutational symmetries, denoting the corresponding sets by $\mathcal{I}$ and $\mathcal{S}$:

1. We call a state permutationally invariant (or just invariant, $\varrho \in \mathcal{I}$) if ϱ is invariant under exchanging the particles. That is, $F\varrho F = \varrho$, where the flip operator is $F = \sum_{ij} |ij\rangle\langle ji|$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry.

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2. We call a state symmetric ($\varrho \in \mathcal{S}$) if it acts on the symmetric subspace only. This is the state space of two d-state bosons.

Clearly, we have $\mathcal{S} \subset \mathcal{I}$.

Expectation value matrix

Definition

Expectation value matrix of a bipartite quantum state is

$$\eta_{kl}(\varrho) := \langle M_k \otimes M_l \rangle_{\varrho},$$

where M_k's are local orthogonal observables for both parties, satisfying

$$\mathrm{Tr}(M_k M_l) = \delta_{kl}.$$

Equivalence of several entanglement conditions for symmetric states

Observation 1. Let $\varrho \in \mathcal{S}$ be a symmetric state. Then the following separability criteria are equivalent:

1. ϱ has a positive partial transpose (PPT), $\varrho^{T_A} \geq 0$.
2. ϱ satisfies the Computable Cross Norm-Realignment (CCNR) criterion, $||R(\varrho)||_1 \leq 1$, where $R(\varrho)$ is the realignment map and $||...||_1$ is the trace norm.
3. $\eta \geq 0$, or, equivalently $\langle A \otimes A \rangle \geq 0$ for all observables A.
4. The correlation matrix, defined via the matrix elements as

$$C_{kl} := \langle M_k \otimes M_l \rangle - \langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle$$

   is positive semidefinite: $C \geq 0$. [A.R. Usha Devi *et al.*, Phys. Rev. Lett. **98**, 060501 (2007).]
5. The state satisfies several variants of the Covariance Matrix Criterion (CMC). Latter are strictly stronger than the CCNR criterion for non-symmetric states.

Proof of Observation 1: Covariance Matrix Criterion

- Variants of the Covariance Matrix Criterion:

$$\|C\|_1^2 \leq [1 - \mathrm{Tr}(\varrho_A^2)][1 - \mathrm{Tr}(\varrho_B^2)]$$

or

$$2\sum |C_{ii}| \leq [1 - \mathrm{Tr}(\varrho_A^2)] + [1 - \mathrm{Tr}(\varrho_B^2)].$$

[O. Gühne *et al.*, PRL **99**, 130504 (2007); O. Gittsovich *et al.*, PRA **78**, 052319 (2008).]

- If ϱ is symmetric, the fact that C is positive semidefinite gives $\|C\|_1 = \mathrm{Tr}(C) = \sum \Lambda_k - \sum_k \mathrm{Tr}(\varrho_A M_k)^2 = 1 - \mathrm{Tr}(\varrho_A^2)$ and similarly, $\sum_i |C_{ii}| = \sum_i C_{ii} = 1 - \mathrm{Tr}(\varrho_A^2)$.
- Hence, a state fulfilling $C \geq 0$ fulfills also both criteria. On the other hand, a state violating $C \geq 0$ must also violate these criteria, as they are strictly stronger than the CCNR criterion

Are there symmetric bound entangled states?

- For symmetric states,
  1. CCNR,
  2. $\eta \geq 0$,
  3. $C \geq 0$ and
  4. CMC

  are equivalent to the PPT criterion.
- It is then quite hard to find symmetric PPT entangled states.

Do symmetric bound entangled states exist at all?


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Symmetric bound entangled states–Bipartite case

Symmetric bound entangled states

- Breuer presented, for even $d \geq 4$, a single parameter family of bound entangled states that are $\mathcal{I}$ symmetric

$$\varrho_{\mathrm{B}} = \lambda|\Psi_0^d\rangle\langle\Psi_0^d| + (1-\lambda)\Pi_s^d.$$

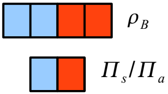
[H.-P. Breuer, PRL **97**, 080501 (2006); see also K.G.H. Vollbrecht and M.M. Wolf, PRL **88**, 247901 (2002).]

- The state is PPT entangled for $0 \leq \lambda \leq 1/(d+2)$. Here $|\Psi_0\rangle$ is the singlet state and Π_s is the normalized projector to the symmetric subspace.

- Idea to construct bound entangled states with an $\mathcal{S}$-symmetry: We embed a low dimensional entangled state into a higher dimensional Hilbert space, such that it becomes symmetric, while it remains entangled.

An 8×8 symmetric bound entangled states

- We consider the symmetric state



$$\hat{\varrho} = \lambda \Pi_a^{d_2} \otimes |\Psi_0^d\rangle\langle\Psi_0^d| + (1-\lambda)\Pi_s^{d_2} \otimes \Pi_s^d.$$

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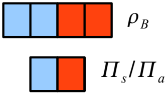

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- If the original system is of dimension $d \times d$ then the system of $\hat{\varrho}$ is of dimension $dd_2 \times dd_2$. Since ϱ_{B} is the reduced state of $\hat{\varrho}$, if the first is entangled, then the second is also entangled.

- For $d_2 = 2$ and $d = 4$, numerical calculation shows that $\hat{\varrho}$ is PPT for $\lambda < 0.062$.

This provides an example of an $\mathcal{S}$ symmetric bound entangled state of size 8×8.

Symmetric bound entangled states–Multipartite case

Symmetric bound entangled state via numerics– Basic idea

- Let us consider an N-qubit symmetric state, that is, a state of the symmetric subspace. We consider even N.

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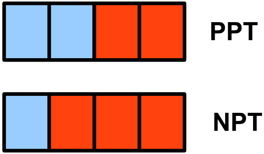
- Thus any state that is PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition while NPT with respect to some other partition is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition.



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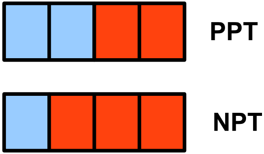

- Since the state is symmetric, it can straightforwardly be mapped to a $(\frac{N}{2}+1) \times (\frac{N}{2}+1)$ bipartite symmetric state.

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- Then, we compute the minimum nonzero eigenvalue of the partial transpose of ϱ with respect to all other partitions

$$\lambda_{\min}(\varrho) := \min_k \min_l \lambda_l(\varrho^{T_{I_k}}).$$

If $\lambda_{\min}(\varrho) < 0$ then the state is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition. If it is non-negative then we start an optimization process for decreasing this quantity.

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- We generate another random density matrix $\Delta\varrho$, and check the properties of

$$\varrho' = (1-\varepsilon)\varrho + \varepsilon\Delta\varrho, \tag{1}$$

  where $0 < \varepsilon < 1$ is a small constant. If ϱ' is still PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition and $\lambda_{\min}(\varrho') < \lambda_{\min}(\varrho)$ then we use ϱ' as our new random initial state ϱ.

3×3 symmetric bound entangled state

- Repeating this procedure, we obtained a four-qubit symmetric state this way

$$\varrho_{BE4} = \begin{pmatrix} 0.22 & 0 & 0 & -0.059 & 0 \\ 0 & 0.176 & 0 & 0 & 0 \\ 0 & 0 & 0.167 & 0 & 0 \\ -0.059 & 0 & 0 & 0.254 & 0 \\ 0 & 0 & 0 & 0 & 0.183 \end{pmatrix}.$$

  The basis states are $|0\rangle := |0000\rangle$, $|1\rangle := \mathrm{sym}(|1000\rangle)$, $|2\rangle := \mathrm{sym}(|1100\rangle), ...$

- The state is bound entangled with respect to the $2:2$ partition. This corresponds to a 3×3 bipartite symmetric bound entangled state, demonstrating the simplest possible symmetric bound entangled state.

Five- and six-qubit fully PPT entangled states

- Our method can be straightforwardly generalized to create multipartite bound entangled states of the symmetric subspace, such that *all* bipartitions are PPT ("fully PPT states").

- We found such a state for five and six qubits.

- Note that these states are both fully PPT and genuine multipartite entangled. It is further interesting to relate this to the Peres conjecture, stating that fully PPT states cannot violate a Bell inequality.


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Conclusions

- In summary, we have discussed entanglement in symmetric systems.
- We showed that for states that are in the symmetric subspace several relevant entanglement conditions, especially the PPT criterion, the CCNR criterion, and the criterion based on covariance matrices matrices, coincide.
- We proved the existence of symmetric bound entangled states, in particular, a 3×3, five-qubit and six-qubit symmetric PPT entangled states.
- See G. Tóth and O. Gühne, arxiv:0812.4453.

*** THANK YOU ***