Extremal properties of the variance and the quantum Fisher information

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Norwegian Sea
Barents Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
Luleå
Oulu
Umeå
Trondheim
White Sea
Arkhangelsk
Lake Onega
Lake Ladoga
Gulf of Bothnia
Tampere
Helsinki
Turku
Gulf of Finland
St. Petersburg
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
Bergen
Oslo
Stockholm
ÅLAND ISLANDS
Tallinn
ESTONIA
Gävle
Stavanger
Rockall (U.K.)
HEBRIDES
North Atlantic Ocean
Aberdeen
Glasgow
Edinburgh
North Sea
Göteborg
Gotland
Baltic Sea
Riga
LATVIA
Smolensk
Vitsyebsk
DENMARK
Copenhagen
Malmö
Öland
LITHUANIA
Vilnius
Kaliningrad
RUSSIA
Mahilyow
Minsk
BELARUS
Belfast
UNITED KINGDOM
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Liverpool
Leeds
Manchester
Birmingham
Cardiff
London
Celtic Sea
Bornholm
Gdańsk
Hrodna
Homyel
Hamburg
Bremen
Berlin
Poznań
Warsaw
POLAND
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Lille
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Rivne
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English Channel
Guernsey (U.K.)
Jersey (U.K.)
Paris
LUX.
Luxembourg
Stuttgart
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SLOVAKIA
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Chernivtsi
Chisinau
Mykolayiv
Strasbourg
Vienna
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HUNGARY
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Ljubljana
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Zagreb
CROATIA
Milan
Venice
Turin
Genoa
BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
SERBIA AND MONTENEGRO
Varna
BULGARIA
Sofia
A Coruña
Bilbao
Bordeaux
Toulouse
MONACO
Andorra la Vella
ANDORRA
Marseille
SAN MARINO
Florence
ITALY
Rome
Adriatic Sea
Ligurian Sea
Corsica
Skopje
F.Y.R.O.M.
Istanbul
Porto
Zaragoza
PORTUGAL
Madrid
Lisbon
SPAIN
Barcelona
Valencia
Balearic Sea
VATICAN CITY
Tirana
ALB.
Thessaloniki
Naples
Tyrrhenian Sea
Sardinia
GREECE
Aegean Sea
Athens
BALEARIC ISLANDS
Sevilla
Gibraltar (U.K.)
Málaga
Strait of Gibraltar
Alborán Sea
Mediterranean Sea
Cagliari
Palermo
Sicily
Ionian Sea
Crete
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Algiers
Tunis
TUNISIA
Valletta
MALTA
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Why variance and the quantum Fisher information are important?

2. **Variance and quantum Fisher information**
   - Basic definitions
   - Entanglement detection with the variance
   - Entanglement detection with the quantum Fisher information

3. **Generalized variance and quantum Fisher information**
   - Generalized variance
   - Generalized quantum Fisher information
   - Generalized quantities in the literature

Why variance and the quantum Fisher information is important?

- Variance appears in all areas of physics.
- Quantum Fisher information is a central notion in metrology.
- Concave roofs, convex roofs are interesting in entanglement theory.

Outline

Variance

- The variance is defined as
$$(\Delta A)^2{}_{\varrho} = \langle A^2 \rangle_{\varrho} - \langle A \rangle_{\varrho}^2.$$

- The variance is concave.

Quantum Fisher information (QFI)

- The parameter θ must be estimated by measuring he output state :

$$\varrho \Rightarrow \boxed{U(\theta) = \exp(-iA\theta)} \Rightarrow \varrho_{output}$$

- Cramér-Rao bound

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q^{\mathrm{usual}}[\varrho, A]}}.$$

- The quantum Fisher information is

$$F_Q^{\mathrm{usual}}[\varrho, A] = 2 \sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- For pure states, $F_Q^{\mathrm{usual}}[\varrho, A] = 4(\Delta A)^2{}_{\varrho}$, and it is convex.

[E.g., P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).]

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Entanglement detection with the variance

- Two properties of the variance are used:
  - For pure states, it is $\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2$.
  - It is concave.
- Any other quantity with these propeties could be used instead of the variance.
- If it were smaller than the variance, then it would even be better than the variance for this purpose.

[O. Gühne, Phys. Rev. Lett. 92, 117903 (2004).]

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Entanglement detection with the QFI

- Two properties of the QFI are used:
  - For pure states, it is $4(\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2)$.
  - It is convex.
- Any other quantity with these propeties could be used instead of the QFI.
- If it were larger than the usual quantum Fisher information, then it would even be better for this purpose.

[L. Pezze and A. Smerzi , Phys. Rev. Lett. 102, 100401 (2009).]

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Generalized variance

Definition 1. Generalized variance $\mathrm{var}_\varrho(A)$ is defined as follows.

1. For pure states, we have
$$\mathrm{var}_\Psi(A) = (\Delta A)^2{}_\Psi.$$
2. For mixed states, $\mathrm{var}_\varrho(A)$ is concave in the state.

Definition 2. The minimal generalized variance $\mathrm{var}_\varrho^{\mathrm{min}}(A)$ is defined as follows.

1. For pure states, it equals the usual variance
$$\mathrm{var}_\Psi^{\mathrm{min}}(A) = (\Delta A)^2{}_\Psi,$$
2. For mixed states, it is defined through a concave roof construction
$$\mathrm{var}_\varrho^{\mathrm{min}}(A) = \sup_{\{p_k,\Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$
where
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Theorem 1

Theorem 1.
The minimal generalized variance is the usual variance

$$\mathrm{var}_{\varrho}^{\min}(A) = (\Delta A)^2{}_{\varrho}.$$

In other words, the variance its own concave roof.

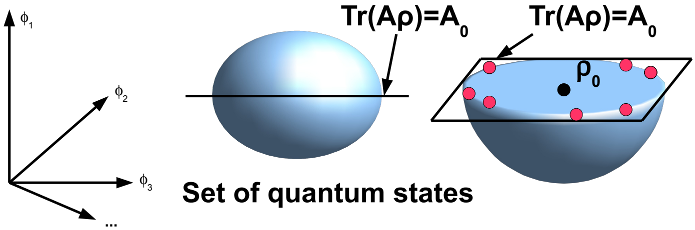
Hand waving proof:

$$(\Delta A)^2_{\varrho} = \sum_k p_k (\Delta A)^2{}_{\Psi_k} + (\langle A \rangle_{\Psi_k} - \langle A \rangle_{\varrho})^2.$$

You can always find a decomposition such that $\langle A \rangle_{\Psi_k} = \langle A \rangle_{\varrho}$ for all k.

Theorem 1

Hand waving proof, continuation; geometric argument:



For details, please see arxiv:1109.2831.

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Generalized quantum Fisher information

Definition 3. Generalized quantum Fisher information $F_Q[\varrho, A]$:

1. For pure states, we have

$$F_Q[\varrho, A] = 4(\Delta A)^2_{\Psi}.$$

   The factor 4 appears for historical reasons.

2. For mixed states, $F_Q[\varrho, A]$ is convex in the state.

Definition 4. Maximal quantum Fisher information $F_Q^{\max}[\varrho, A]$:

1. For pure states, it equals four times the usual variance

$$F_Q^{\max}[\varrho, A] = 4(\Delta A)^2_{\Psi}.$$

2. For mixed states, it is defined through a convex roof construction

$$F_Q^{\max}[\varrho, A] = 4 \inf_{\{p_k, \Psi_k\}} p_k (\Delta A)^2_{\Psi_k}.$$

Theorem 2

Theorem 2.

For rank-2 states

$$F_Q^{\max}[\varrho, A] = F_Q^{\mathrm{usual}}[\varrho, A].$$

For an analytic proof, see G. Tóth and D. Petz, arxiv:1109.2831.

In other words, the quantum Fisher information is four times the convex roof of the variance for rank-2 states.

Numerics for rank>2

- The maximal generalized q. Fisher information can be written as

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k \langle A\rangle_{\Psi_k}^2\right).$$

- Rewriting the term quadratic in expectation values as an operator acting on a bipartite system

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k \langle A \otimes A\rangle_{\Psi_k\otimes\Psi_k}\right).$$

- Further transformations lead to

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k,|\Psi_k\rangle\}} \langle A \otimes A\rangle_{\sum_k p_k |\Psi_k\rangle\langle\Psi_k|^{\otimes 2}}\right).$$

Numerics for rank>2 II

- Hence we obtain that

$$F_Q^{\max}[\varrho, A]. = 4\left(\langle A^2\rangle_\varrho - \sup_{\substack{\varrho_{ss}\in S_s, \\ \mathrm{Tr}_1(\varrho_{ss})=\varrho}} \langle A\otimes A\rangle_{\varrho_{ss}}\right),$$

  where S_s are symmetric separable states.

- Instead of the separable states, we can do the optimization for PPT states or states with a PPT symmetric extension.

- Extensive numerics on random ϱ and A confirm that

$$F_Q = F_Q^{max}$$

  holds within a large degree of accuracy.

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Generalized variance and quantum Fisher information in the literature

- Generalized variances and quantum Fisher informations of D. Petz.
- Defines a variance and a corresponding quantum Fisher information for each standard matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$.
- *Surprisingly*, his variances and quantum Fisher information definitions fit the definitions of this presentation.
- Our quantities are extremal even within the sets defined by Petz et al. However, our definitions are broader.

D. Petz, *Quantum Information Theory and Quantum Statistics* (Springer, 2008).

D. Petz, J. Phys. A: Math. Gen. **35**, 79 (2003).

P. Gibilisco, F. Hiai and D. Petz, IEEE Trans. Inform. Theory **55**, 439 (2009).

F. Hiai and D. Petz, From quasi-entropy, http://arxiv.org/abs/1009.2679.

Conjecture

Conjecture

We conjecture that

$$F_Q = F_Q^{max}$$

for density matrices of any rank and for any Hermitian$A$.

Conjecture based on

- Analytics for rank 2
- Extensive numerics for rank>2
- Statement is true for a large subset

Follow-up

- Proof: Sixia Yu, arxiv 1302.5311.
- We should look for connections to
  [B.M. Escher,R.L.de Matos Filho, and L. Davidovich, Nature Phys. (2011)] .

Summary

- We defined the generalized variance and the generalized quantum Fisher information.
- We found that the variance is its own concave roof, while the quantum Fisher information is its own convex roof.

See:
G. Tóth and D. Petz,
Extremal properties of the variance and the quantum Fisher information, Phys. Rev. A, in press;
arxiv:1109.2831.

THANK YOU FOR YOUR ATTENTION!

erc European Research Council

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EUSKO JAURLARITZA GOBIERNO VASCO