Detection of multipartite entanglement close to symmetric Dicke states

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Jan Mayen (NORWAY)
Greenland Sea
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Barents Sea
Norwegian Sea
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Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
Luleå
Oulu
Umeå
White Sea
Arkhangelsk
Lake Onega
Lake Ladoga
Trondheim
Gulf of Bothnia
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Gulf of Finland
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Cagliari
Palermo
Sicily
Ionian Sea
Tunis
TUNISIA
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles
Boundary representation not necessarily authoritative

Outline

1. **Motivation**
   - Why multipartite entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Spin squeezing for Dicke states**
   - Entanglement detection close to Dicke states
   - Detection of multipartite entanglement close to Dicke states
   - Our conditions are stronger than the original conditions

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and G. Tóth, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

The standard spin-squeezing criterion

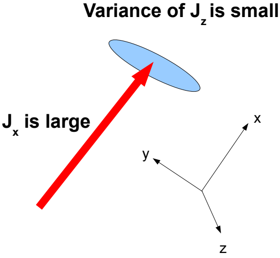
- The spin squeezing criteria for entanglement detection is

$$\xi_{\rm s}^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

If $\xi_{\rm s}^2 < 1$ then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States detected are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \le \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \le (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \ge \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

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Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \le (N-1)(\Delta J_m)^2 + \frac{N}{2}.$$

- It detects states close to symmetric Dicke states with $\langle J_z \rangle = 0$ defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

since for these states we have

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$
$$\langle J_z^2 \rangle = 0.$$

Dicke states

Based on the above inequality, let us define a new spin squeezing parameter

$$\xi_{\rm os}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2\rangle - \frac{N}{2}}.$$

[G. Vitagliano, I. Apellaniz, I.L. Egusquiza, and GT, PRA (2014)]

- For the symmetric Dicke state with $\langle J_z\rangle = 0$, the numerator is minimal, the denominator is maximal.

- The original spin squeezing parameter would not detect the Dicke state as entangled, since

$$\xi_{\rm s}^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2} = N\frac{(\Delta J_z)^2}{0}.$$

Fully polarized states

- Relation between the second moments and the expectation value

$$\langle J_x^2 \rangle = \langle J_x \rangle^2 + (\Delta J_x)^2 \geq \langle J_x \rangle^2.$$

- For states polarized in the x-direction and spin squeezed along the z-direction, for $N \gg 1$, we have

$$\langle J_x^2 \rangle \approx \langle J_x \rangle^2 \gg N.$$

Hence, for fully polarized states

$$\xi_{\rm os}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2 \rangle - \frac{N}{2}} \approx \xi_{\rm s}^2 = N\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{n=1}^{M} |\psi^{(n)}\rangle,$$

where $|\psi^{(n)}\rangle$ is the state of at most k qubits.

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: C. Gross, T. Zibold, E. Nicklas, J. Esteve, M. K. Oberthaler, Nature 464, 1165 (2010).]

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

  and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.

Multipartite entanglement around Dicke states II

- Sørensen-Mølmer condition for *k*-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

which is true for pure *k*-producible states.

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

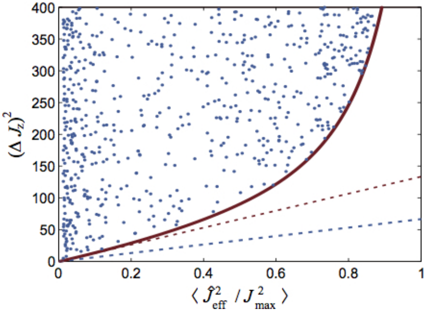
Multipartite entanglement around Dicke states III

- For large N, and $k \ll N$ we have

$$(\Delta J_z)^2 \gtrsim J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle}}{J_{\max}} \right).$$

Concrete example

- Let us draw the boundary of k-producible states.



- For $N = 8000$ particles, state below the curve have a larger than 28-particle entanglement.
- The blue dashed line is the condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- The red dashed line is the tangent of our curve.

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Our condition is stronger

- Examine, when our spin squeezing parameter is stronger:

$$\xi_{\mathrm{os}}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2\rangle - \frac{N}{2}} < \xi_{\mathrm{s}}^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

- Noisy states of the form

$$\varrho_{\mathrm{noisy}} = (1-p)\varrho + p\frac{\mathbb{1}}{2^N}.$$

- For this state,

$$\left(\langle J_x^2 + J_y^2\rangle_{\mathrm{noisy}} - \tfrac{N}{2}\right) = \color{red}{(1-p)}\left(\langle J_x^2 + J_y^2\rangle - \tfrac{N}{2}\right),$$
$$\left(\langle J_x\rangle^2 + \langle J_y\rangle^2\right)_{\mathrm{noisy}} = \color{red}{(1-p)^2}\left(\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2\right).$$

- Hence, $\xi_{\mathrm{os}}^2 < \xi_{\mathrm{s}}^2$ if

$$(\Delta J_x)^2 + (\Delta J_y)^2 > \tfrac{N}{2} - p\left(\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2\right).$$

  Thus, in all practical cases our relation is stronger for large N: fully polarized states with $\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2 > O(N)$ and Dicke states.

Our condition is stronger II

- We can also incorporate the original spin squeezing parameter using

$$\left(\langle J_x\rangle^2 + \langle J_y\rangle^2\right) = \frac{1}{\xi_{\rm s}^2} N (\Delta J_z)^2. \tag{1}$$

- Hence, $\xi_{\rm os}^2 < \xi_{\rm s}^2$ if

$$(\Delta J_x)^2 + (\Delta J_y)^2 > N\left(\frac{1}{2} - p\frac{(\Delta J_z)^2}{\xi_{\rm s}^2}\right).$$

- Assuming $\xi_{\rm s} < 1$, the right-hand side is negative for $p > 0$ unless we have $(\Delta J_z)^2 \sim O(N^0)$. Not realistic.

Hence, for large N, if $\xi_{\rm s} < 1$ then (to a very good degree of approximation)

$$\xi_{\rm os}^2 \leq \xi_{\rm s}^2.$$

[G. Vitagliano, I. Apellaniz, I.L. Egusquiza, and GT, PRA (2014)]

Our condition is stronger - multipartite case

- Our entanglement condition is stronger if

$$\langle J_x^2 + J_y^2 \rangle - J_{\max}(\tfrac{k}{2} + 1) \geq \langle J_x \rangle^2 + \langle J_y \rangle^2.$$

- Noisy states of the form

$$\varrho_{\text{noisy}} = (1-p)\varrho + p\frac{\mathbb{1}}{2^N}.$$

- Our entanglement condition is stronger if

$$(\Delta J_x)^2 + (\Delta J_y)^2 \geq \tfrac{N}{2}(\tfrac{k}{2} + 1) - p\left(\langle J_x \rangle_\varrho^2 + \langle J_y \rangle_\varrho^2\right).$$

- Thus, in all practical cases our relation is stronger for large N :
  - fully polarized states with $\langle J_x \rangle_\varrho^2 + \langle J_y \rangle_\varrho^2 \sim O(N^q)$ with $q > 1$,
  - Dicke states with $(\Delta J_x)^2 + (\Delta J_y)^2 \sim O(N^2)$.
- Similar argument, as before for $\xi_s < 1$.

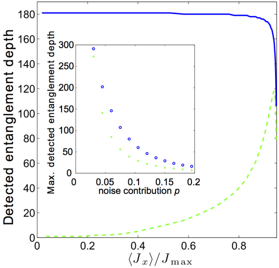
Our condition is stronger - multipartite case II

- Consider spin squeezed states as ground states of

$$H(\Lambda) = J_z^2 - \Lambda J_x.$$

For $\Lambda = \infty$, the ground state is fully polarized. For $\Lambda = 0$, it is the symmetric Dicke state.

- Our condition vs. original condition for N=4000 and p=0.05



Summary

- We showed how to detect multipartite entanglement close to Dicke states.
- We need to measure collective quantities only.
- The condition is optimal: it detects all entangled states that can be detected based on the measured quantities.

See:

G. Vitagliano, I Apellaniz, I.L. Egusquiza, and G. Tóth, PRA (2014).

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, PRL, in press.

THANK YOU FOR YOUR ATTENTION!

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