Spin-squeezing inequalities for entanglement detection in cold gases Phys. Rev. Lett. 107, 240502 (2011)

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11 June 2013


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangel'sk
Luleå
Oulu
Umeå
Trondheim
Gulf of Bothnia
Tampere
Lake Onega
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
HEBRIDES
Rockall (U.K.)
Bergen
Oslo
Stockholm
Gävle
Turku
Helsinki
ALAND ISLANDS
Gulf of Finland
St. Petersburg
Tallinn
ESTONIA
Stavanger
Göteborg
Gotland
Riga
LATVIA
North Atlantic Ocean
North Sea
Aberdeen
Glasgow
Edinburgh
UNITED KINGDOM
Belfast
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IRELAND
Irish Sea
Isle of Man (U.K.)
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Jersey (U.K.)
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
Bornholm
Kaliningrad
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Belgrade
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Varna
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Sofia
Istanbul
A Coruña
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Bordeaux
Toulouse
Andorra la Vella
ANDORRA
MONACO
Marseille
SAN MARINO
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Ligurian Sea
ITALY
Rome
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SPAIN
Valencia
Balearic Sea
Corsica
VATICAN CITY
Naples
Sardinia
Tyrrhenian Sea
Sevilla
BALEARIC ISLANDS
Cagliari
Ionian Sea
Gibraltar (U.K.)
Málaga
Mediterranean Sea
Palermo
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
TUNISIA
Rabat
Casablanca
MOROCCO
ALGERIA
Valletta
MALTA
Crete
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Why spin squeezing inequalities for $j > \frac{1}{2}$ is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Most experiments use atoms with $j > \frac{1}{2}$.

Genuine multipartite entanglement

Definition

A state is (fully) separable if it can be written as $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

k-producibility/*k*-entanglement

Definition

A pure state is *k*-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most k qubits. A mixed state is k-producible, if it is a mixture of k-producible pure states.
[O. Gühne and G. Tóth, New J. Phys 2005.]

- In many-particle systems, this is the only meaningful characterization of entanglement.
- That is, genuine multipartite entanglement is very difficult to detect in such systems.

Outline

Physical systems

State-of-the-art in experiments

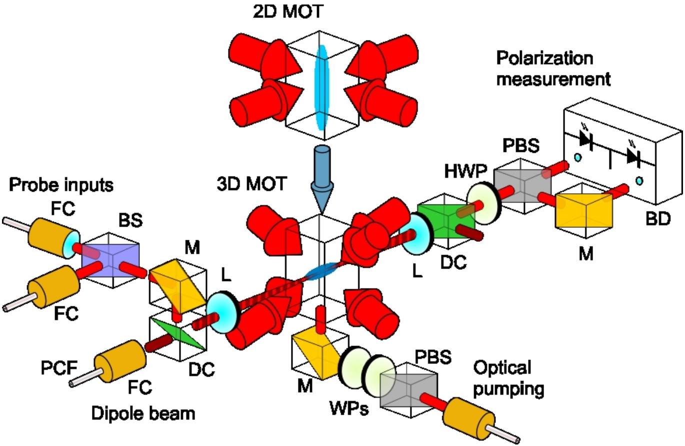
- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Physical systems II

For example: Spin squeezing in a cold atomic ensemble (not BEC!)



Picture from M.W. Mitchell, ICFO, Barcelona.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

  where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

Squeezing

- The variances of the two quadrature components are bounded

$$(\Delta x)^2(\Delta p)^2 \geq const.$$

- Coherent states saturate the inequality.
- Squeezed states are the states for which one of the quadrature components have a smaller variance than for a coherent state.
- Can one use similar ideas for spin systems?

Outline

Spin squeezing

- The variances of the angular momentum components are bounded

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

  If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{|\langle Jz\rangle|}{2}$ then the state is called spin squeezed.

- z is the direction of the mean spin!

- The angular momentum of such a state has a small variance in one direction.

- The variance is large in an orthogonal direction.

[M. Kitagawa and M. Ueda, PRA 47, 5138 (1993).]

Outline

The standard spin-squeezing criterion

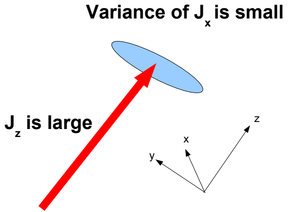
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

- If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States violating it are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled.

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4}, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2}, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},
\end{aligned}$$

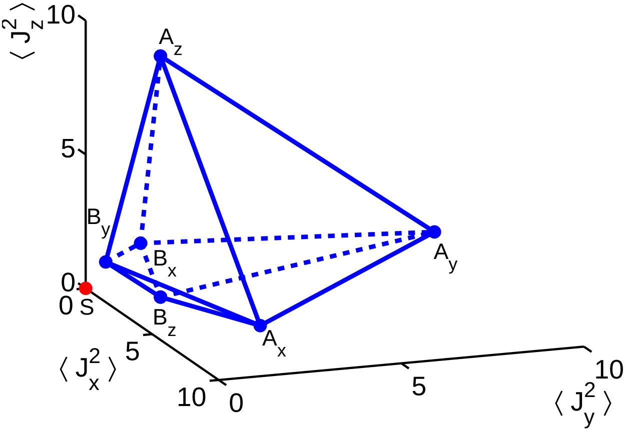
where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

Generalized spin squeezing criteria for $j = \frac{1}{2}$

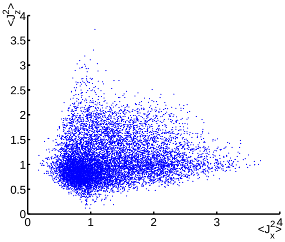
- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space. The polytope has six extreme points: $A_{x/y/z}$ and $B_{x/y/z}$.

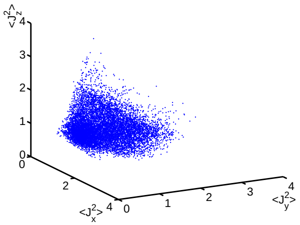
- For $\left\langle \vec{J} \right\rangle = 0$ and $N = 6$ the polytope is the following:



Completeness

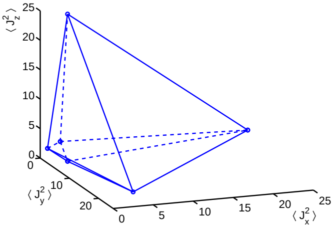
- Random separable states:

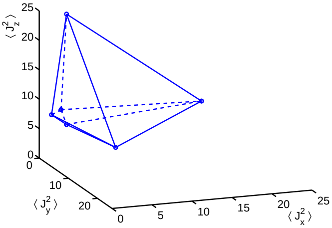




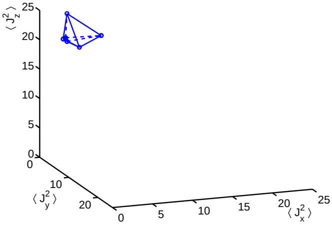
- The completeness can be proved for large N.

- The polytope for $N = 10$ and
  $J = (0, 0, 0)$, $J = (0, 0, 2.5)$,





and $J = (0, 0, 4.5)$.



Outline

Basic ideas for the $j > \frac{1}{2}$ case

- Particles with d>2 internal states.
- a_k for $k = 1, 2, ..., M$ denote single-particle operators with the property

$$\mathrm{Tr}(a_k a_l) = C\delta_{kl},$$

where C is a constant.

- We need the upper bound K for the inequality

$$\sum_{k=1}^{M} \langle a_k^{(n)} \rangle^2 \leq K.$$

- The N-qudit collective operators used in our criteria will be denoted by

$$A_k = \sum_n a_k^{(n)}.$$

“Modified” quantities for $j > \frac{1}{2}$

- For the $j = \frac{1}{2}$ case, the SSIs were developed based on the first and second moments and variances of the such collective operators.

- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{A}_k^2 \rangle := \langle A_k^2 \rangle - \langle \sum_n (a_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle a_k^{(n)} a_k^{(m)} \rangle$$

  and the modified variance

$$(\tilde{\Delta} A_k)^2 := (\Delta A_k)^2 - \langle \sum_n (a_k^{(n)})^2 \rangle.$$

- These are essential to get tight equations for $j > \frac{1}{2}$.

Basic equation

- For separable states, i.e., for states that can be written as a mixture of product states,

$$(N-1)\sum_{k\in I}(\tilde{\Delta}A_k)^2 - \sum_{k\notin I}\langle(\tilde{A}_k)^2\rangle \geq -N(N-1)K$$

  holds, where each index set $I \subseteq \{1, 2, ..., M\}$ defines one of the 2^M inequalities.

- Note that $I = \emptyset$ and $I = \{1, 2, ..., M\}$ are among the possibilities.

Derivation

- We consider product states of the form $|\Phi\rangle = \otimes_n |\phi_n\rangle$. For such states, we have $(\Delta \tilde{A}_k)^2{}_\Phi = -\sum_n \langle a_k^{(n)} \rangle^2$.
- Hence, the left-hand side of the inequality equals

$$
\begin{aligned}
&-\sum_n (N-1) \sum_{k \in I} \langle a_k^{(n)} \rangle^2 - \sum_{k \notin I} \left(\langle A_k \rangle^2 - \sum_n \langle a_k^{(n)} \rangle^2 \right) \\
&\geq -\sum_n (N-1) \sum_{k=1}^{M} \langle a_k^{(n)} \rangle^2 \geq -N(N-1)K
\end{aligned}
$$

- We used that $\langle A_k \rangle^2 \leq N \sum_n \langle a_k^{(n)} \rangle^2$.
- The equation is saturated by all states of the form $|\phi\rangle^{\otimes N}$.

Outline

The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates

- Application 1:

$$a_k = \{j_x, j_y, j_z\}.$$

- For spin-j particles for $j > \frac{1}{2}$, we can measure the collective angular momentum operators:

$$J_l := \sum_{k=1}^{N} j_l^{(k)},$$

where $l = x, y, z$ and $j_l^{(k)}$ are the angular momentum coordinates [i.e., SU(2) generators].

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$$

variances.

Remember: “Modified” quantities for $j > \frac{1}{2}$

- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{J}_k^2 \rangle := \langle J_k^2 \rangle - \langle \sum_n (j_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle j_k^{(n)} j_k^{(m)} \rangle$$

  and the modified variance

$$(\tilde{\Delta} J_k)^2 := (\Delta j_k)^2 - \langle \sum_n (j_k^{(n)})^2 \rangle.$$

- These are essential to get tight equations for $j > \frac{1}{2}$.

The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates II

- For fully separable states of spin-j particles, all the following inequalities are fulfilled

$$
\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq Nj(Nj+1), \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj, \\
\langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N-1)j^2 &\leq (N-1)(\tilde{\Delta} J_m)^2, \\
(N-1)\left[(\tilde{\Delta} J_k)^2 + (\tilde{\Delta} J_l)^2\right] &\geq \langle \tilde{J}_m^2 \rangle - N(N-1)j^2,
\end{aligned}
$$

  where k, l, m take all possible permutations of x, y, z.

- Violation of any of the inequalities implies entanglement.

Completeness

- In the large N limit, the inequalities with the angular momentum are complete.
- That is, it is not possible to come up with a new entanglement conditions with based on $\langle J_k \rangle$ and seed on $\langle J_k \rangle$ and $\langle \tilde{J}_k^2 \rangle$ that detect states not detected by these inequalities.

Mapping qubit inequalities to qudits

- Take an inequality valid for N-qubit separable states of the form

$$f(\{\langle J_l \rangle\}_{l=x,y,z}, \{\langle \tilde{J}_l^2 \rangle\}_{l=x,y,z}) \geq \text{const.}$$

  All of the generalized SSIs have this form.

- An entanglement condition can be transformed to a criterion for a system of N spin-j particles by the substitution

$$\langle J_l \rangle \quad \to \quad \frac{1}{2j}\langle J_l \rangle, \qquad \langle \tilde{J}_l^2 \rangle \to \frac{1}{4j^2}(\langle \tilde{J}_l^2 \rangle).$$

The usual spin squeezing inequality for $j > \frac{1}{2}$

- The standard spin-squeezing inequality becomes

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} + \frac{\sum_n (j^2 - \langle (j_x^{(n)})^2\rangle)}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

  This inequality is violated only if there is entanglement between the spin-j particles.

- Due to the second, nonnegative term on the left-hand side, for $j > \frac{1}{2}$ there are states that violate the original inequality, but do not violate this one.

- Thus, there is spin squeezing without entanglement between the particles.

- Our spin squeezing inequalities are strictly stronger than the original inequality.

Outline

The inequalities for $j > \frac{1}{2}$ with the G_k's

- Application 2:

$$a_k = SU(d) \text{ generators.}$$

- For spin-j particles for $j > 1/2$, we can measure the collective operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

where $l = 1, 2, ..., d^2 - 1$ and $g_l^{(k)}$ are the SU(d) generators.

- We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.

The inequalities for $j > \frac{1}{2}$ with the G_k's

- For a system of d-dimensional particles, we can define collective operators based on the SU(d) generators $\{g_k\}_{k=1}^M$ with $M = d^2 - 1$ as $G_k = \sum_{n=1}^N g_k^{(n)}$.

- The SSIs for G_k have the general form

$$(N-1)\sum_{k\in I}(\tilde{\Delta} G_k)^2 - \sum_{k\notin I}\langle(\tilde{G}_k)^2\rangle \geq -2N(N-1)\frac{(d-1)}{d}.$$

  For instance, for the $d = 3$ case, the SU(d) generators can be the eight Gell-Mann matrices.

- I is a subset of indices between 1 and M. We have 2^M equations!

Outline

One of the generalized spin squeezing criteria

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin,
arXiv:1104.3147.]

Maximally violating states

- For $N = d$, the multipartite SU(d) singlet state maximally violates the condition with $\sum_k (\Delta G_k)^2 = 0$.
- For $N < d$, there is no quantum states for which $\sum_k (\Delta G_k)^2 = 0$.
- This can be seen as follows. It is not possible to create a completely antisymmetric state of d-state particles with less than d particles.

The criterion

A condition for two-producibility for N qudits of dimension d is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

Noise tolerance

- Let us consider SU(d) singlet states (i.e., states with $\langle G_k^2 \rangle = 0$) mixed with white noise as

$$\varrho_{\rm noisy} = (1 - p_{\rm noise})\varrho_{\rm singlet} + p_{\rm noise}\frac{1}{d^N}\mathbb{1}.$$

- Direct calculation shows that such a state is detected as entangled if

$$p_{\rm noise} < \frac{d}{d+1}.$$

  Thus, the noise tolerance in detecting SU(d) singlets is increasing with d!

Advantages of criteria for $j > \frac{1}{2}$

- Most atoms have $j > \frac{1}{2}$. No need to create spin-1/2 subsystems artificially
- Manipulation is possible with magnetic fields rather than with lasers.
- New experiments can be proposed.

Group

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Giuseppe Vitagliano	Ph.D. Student
Iagoba Apellaniz	Ph.D. Student

- Topics
  - Multipartite entanglement and its detection
  - Metrology, cold gases
  - Collaborating on experiments:
    - Weinfurter group, Munich, (photons)
    - Mitchell group, Barcelona, (cold gases)
- Funding:
  - European Research Council starting grant 2011-2016, 1.3 million euros
  - CHIST-ERA QUASAR collaborative EU project (H. Weinfurter)
  - Grants of the Spanish Government and the Basque Government

Summary

- We presented a full set of generalized spin squeezing inequalities with the angular momentum coordinates for $j > \frac{1}{2}$.
- We presented a large set of inequalities with the other collective operators that can be measured.
- These might make possible new experiments and make existing experiments simpler.

See: G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Phys. Rev. Lett. 107, 240502 (2011) + manuscript in preparation.

THANK YOU FOR YOUR ATTENTION!