Spin squeezing and entanglement

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Outline

1. Motivation
2. Entanglement detection with collective observables
3. Optimal spin squeezing inequalities
4. Multipartite bound entanglement in spin models

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Motivation

- In many quantum control experiments the qubits cannot be individually accessed. We still would like to detect entanglement.

- The spin squeezing criterion is already known. Are there other similar criteria that detect entanglement with the first and second moments of collective observables?

Outline

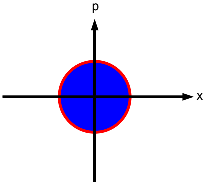
From squeezing to spin squeezing

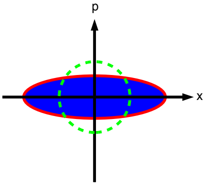
- The variances of the two quadrature components are bounded

$$(\Delta x)^2(\Delta p)^2 \geq const.$$

- Coherent states saturate the inequality.

- Squeezed states are the states for which one of the quadrature components have a smaller variance than for a coherent state.





- Can one use similar ideas for spin systems?

Spin squeezing

Definition

The variances of the angular momentum components are bounded

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2,$$

where the mean spin points into the z direction. If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{|\langle J_z\rangle|}{2}$ then the state is called spin squeezed.

- In practice this means that the angular momentum of the state has a small variance in one direction, while in an orthogonal direction the angular momentum is large.

  [M. Kitagawa and M. Ueda, PRA **47**, 5138 (1993).]

Definition: Entanglement

Definition

Fully separable states are states that can be written in the form

$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes ... \otimes \rho_l^{(N)},$$

where $\sum_l p_l = 1$ and $p_l > 0$.

Definition

A state is entangled if it is not separable.

- Note that one could also look for other type of entanglement in many-particle systems, e.g., entanglement in the two-qubit reduced density matrix.

Definition: Collective quantities

- What if we cannot address the particles individually? This is expected to occur often in future experiments.

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices. We can also measure the $(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$ variances.

The standard spin-squeezing criterion

Definition

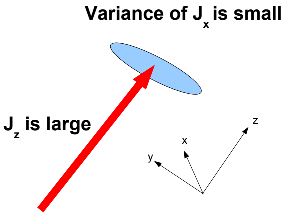
The spin squeezing criterion for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature **409**, 63 (2001).]

- States violating it are like this:



Generalized spin squeezing entanglement criteria I

Separable states must fulfill

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}.$$

It is maximally violated by a many-body singlet, e.g., the ground state of an anti-ferromagnetic Heisenberg chain.

[GT, PRA **69**, 052327 (2004).]

- For such a state
$$\langle J_k^m \rangle = 0.$$
- Note that there are very many states giving zero for the left hand side. The mixture of all such states also maximally violates the criterion.
- Note that a similar inequality works also for a lattice of spins larger than $\frac{1}{2}$. [GT, PRA **69**, 052327 (2004).]

Generalized spin squeezing entanglement criteria II

For states with a separable two-qubit density matrix

$$\left(\langle J_k^2\rangle + \langle J_l^2\rangle - \frac{N}{2}\right)^2 + (N-1)^2\langle J_m\rangle^2 \leq \langle J_m^2\rangle + \frac{N(N-2)}{4}$$

holds.

[J. Korbicz, I. Cirac, M. Lewenstein, PRL **95**, 120502 (2005).]

- Detects all symmetric two-qubit entangled states; can be used to detect symmetric Dicke states.
- Used in ion trap experiment.

  [J. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C.F. Roos, R. Blatt, PRA 74, 052319 (2005).]

Generalized spin squeezing entanglement criteria III

For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{N(N+1)}{4}$$

holds. [GT, J. Opt. Soc. Am. B **24**, 275 (2007).]

- This can be used to detect entanglement close to N-qubit symmetric Dicke states with $\frac{N}{2}$ excitations. For such a state

$$\begin{aligned} \langle J_k \rangle &= 0, \\ \langle J_z^2 \rangle &= 0, \\ \langle J_{x/y}^2 \rangle &= \frac{N(N+2)}{8}. \end{aligned}$$

- For $N = 4$, this state looks like

$$|\Psi\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle).$$

This was realized with photons.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Optimal spin squeezing inequalities

- Let us assume that for a system we know only

$$\mathbf{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle),$$
$$\mathbf{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle).$$

where k, l, m take all the possible permutations of x, y, z.

Definition (Optimal spin squeezing inequalities)

Any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\leq \tfrac{N(N+2)}{4}, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \tfrac{N}{2}, \\
\langle J_k^2\rangle + \langle J_l^2\rangle &\leq (N-1)(\Delta J_m)^2 + \tfrac{N}{2}, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2\rangle + \tfrac{N(N-2)}{4}.
\end{aligned}$$

[GT, C. Knapp, O. Gühne, és H.J. Briegel, PRL 99, 250405 (2007); quant-ph/0702219.]

Derivation of the equations

- Criterion 2

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$

Proof: For product states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = \sum_k (\Delta j_x^{(k)})^2 + (\Delta j_y^{(k)})^2 + (\Delta j_z^{(k)})^2 \geq \frac{N}{2}.$$

It is also true for separable states due to the convexity of separable states.

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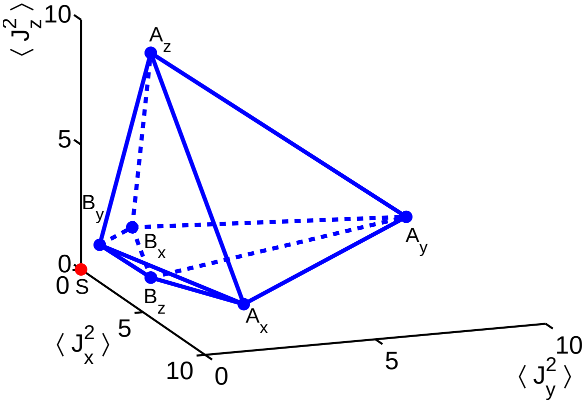
It is also true for separable states due to the convexity of separable states.
- Criterion 3
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \tfrac{N}{2},$$
Proof: For product states
$$(N-1)(\Delta J_x)^2 + \tfrac{N}{2} - \langle J_y^2 \rangle - \langle J_z^2 \rangle = (N-1)\left(\tfrac{N}{4} - \tfrac{1}{4}\sum_k x_k^2\right)$$
$$-\tfrac{1}{4}\sum_{k \neq l} y_k y_l + z_k z_l = ... \geq 0.$$
Here $x_k = \langle \sigma_x^{(k)} \rangle$ and we have to use $(\sum_k s_k)^2 \leq N \sum_k s_k$.

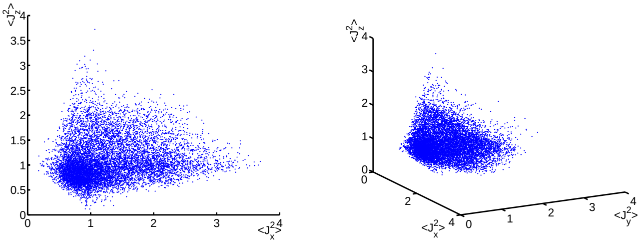
The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z}\rangle$, describe a polytope in the $\langle J^2_{x/y/z}\rangle$ space. The polytope has six extreme points: $A_{x/y/z}$ and $B_{x/y/z}$.
- For $\langle \mathbf{J}\rangle = 0$ and $N = 6$ the polytope is the following:



The polytope II: Numerics

- Random separable states:



The polytope III: Extreme points

The coordinates of the extreme points are

$$A_x := \left[\frac{N^2}{4} - \kappa(\langle J_y\rangle^2 + \langle J_z\rangle^2), \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

$$B_x := \left[\langle J_x\rangle^2 + \frac{\langle J_y\rangle^2 + \langle J_z\rangle^2}{N}, \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

where $\kappa := (N-1)/N$. The points $A_{y/z}$ and $B_{y/z}$ can be obtained from these by permuting the coordinates.

- Now it is easy to prove that an inequality is a necessary condition for separability: All the six points must satisfy it.

The polytope IV: Separable states fill the polytope

- Let us take the $\langle \mathbf{J} \rangle = 0$ case first.
- Then the state corresponding to A_x is the equal mixture of

$$|+1,+1,+1,+1,...\rangle_x$$

  and

$$|-1,-1,-1,-1,...\rangle_x.$$

- The state corresponding to B_x is

$$|+1\rangle_x^{\otimes\frac{N}{2}} \otimes |-1\rangle_x^{\otimes\frac{N}{2}}.$$

- Separable states corresponding to $A_{y/z}$ and $B_{y/z}$ are defined similarly.

The polytope V

- General case: $\langle \mathbf{J} \rangle \neq 0$.
- A separable state corresponding to A_x is

$$\rho_{A_x} = p(|\psi_+\rangle\langle\psi_+|)^{\otimes N} + (1-p)(|\psi_-\rangle\langle\psi_-|)^{\otimes N}.$$

Here $|\psi_{+/-}\rangle$ are the single qubit states with Bloch vector coordinates $(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = (\pm c_x, 2\langle J_y\rangle/N, 2\langle J_z\rangle/N)$ where $c_x := \sqrt{1 - 4(\langle J_y\rangle^2 + \langle J_z\rangle^2)/N^2}$. The mixing ratio is defined as $p := 1/2 + \langle J_x\rangle/(Nc_x)$.

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- If $N_1 := Np$ is an integer, we can also define the state corresponding to the point B_x as

$$|\phi_{B_x}\rangle = |\psi_+\rangle^{\otimes N_1} \otimes |\psi_-\rangle^{\otimes (N-N_1)}.$$

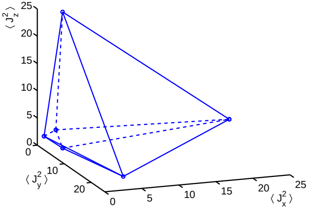
  If N_1 is not an integer then one can find a point B'_x such that such that its distance from B_x is smaller than $\frac{1}{4}$.

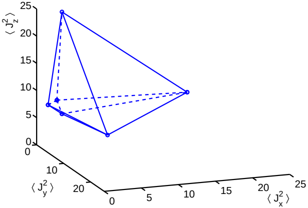
In what sense is the characterization complete?

- For any value of **J** there are separable states corresponding to $A_{x/y/z}$.
- For certain values of **J** and N (e.g., $\mathbf{J} = 0$ and even N) there are separable states corresponding to points $B_{x/y/z}$.
- However, there are always separable states corresponding to points $B'_{x/y/z}$ such that their distance from $B_{x/y/z}$ is smaller than $\frac{1}{4}$.
- In the limit $N \to \infty$ for a fixed normalized angular momentum $\frac{\mathbf{J}}{N/2}$ the sides of the polytope grow as N^2.
- The relative difference between the volume of our polytope and the volume of set of points corresponding to separable states decreases with N as N^{-2}, hence in the macroscopic limit the characterization is complete.

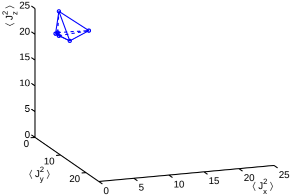
Polytope for various values for J

- The polytope for $N = 10$ and
  $J = (0, 0, 0)$, $J = (0, 0, 2.5)$,





and $J = (0, 0, 4.5)$.

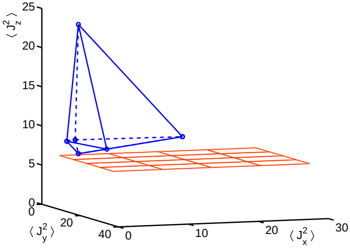


Our inequalities vs. the standard spin squeezing criterion

The standard spin squeezing criterion

$$\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2} \geq \frac{1}{N}$$

is satisfied by all points A_k and B_k, for B_z even equality holds.



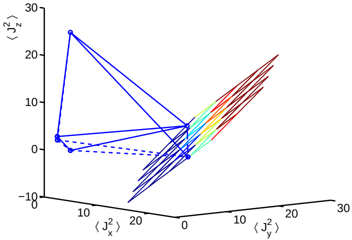
- Polytope for $N = 10$ and $J = (1.5, 0, 2.5)$. States that are detected by the standard criterion are below the red plane.

Our inequalities vs. the Korbicz-Cirac-Lewenstein inequalities

For states with a separable two-qubit density matrix

$$\left(\langle J_k^2\rangle + \langle J_l^2\rangle - \frac{N}{2}\right)^2 + (N-1)^2\langle J_m\rangle^2 \leq \langle J_m^2\rangle + \frac{N(N-2)}{4}$$

holds. [J. Korbicz *et al.* PRL **95**, 120502 (2005).]



- Polytope for $N = 10$ and $J = (0, 0, 0)$. States that are detected by the KCL criterion are below the plane. The plane contains two of the three A_k points.

Correlation matrix

- Our inequalities can be reexpressed with the correlation matrix.
- Basic definitions:

$$\begin{aligned} C_{kl} &:= \tfrac{1}{2}\langle J_k J_l + J_l J_k\rangle, \\ \gamma_{kl} &:= C_{kl} - \langle J_k\rangle\langle J_l\rangle. \end{aligned}$$

- With them we define the interesting quantity

$$\mathfrak{X} := (N-1)\gamma + C.$$

Correlation matrix II

- Now we can rewrite our inequalities as

$$
\begin{aligned}
\mathrm{Tr}(\mathfrak{X}) &\leq \frac{N^2(N+2)}{4} - (N-1)|\mathbf{J}|^2, \\
\mathrm{Tr}(\mathfrak{X}) &\geq \frac{N^2}{2} + |\mathbf{J}|^2, \\
\lambda_{\min}(\mathfrak{X}) &\geq \frac{1}{N}\mathrm{Tr}(\mathfrak{X}) + \frac{N-1}{N}|\mathbf{J}|^2 - \frac{N}{2}, \\
\lambda_{\max}(\mathfrak{X}) &\leq \frac{N-1}{N}\mathrm{Tr}(\mathfrak{X}) - \frac{N-1}{N}|\mathbf{J}|^2 - \frac{N(N-2)}{4},
\end{aligned}
$$

For fixed $|\mathbf{J}|$ these equations describe a polytope in the space of the three eigenvalues of $\mathfrak{X}$.

- These new inequalities detect all entangled quantum states that can be detected based on knowing the correlation matrix and **J**.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, arXiv:0806.1048.]

Outline

Two-qubit entanglement

- Our criteria can detect entangled states for which the reduced two-qubit density matrix is separable.

- This might look surprising since all our criteria contain operator expectation values that can be computed knowing the average two-qubit density matrix

$$\rho_{12} := \frac{1}{N(N-1)} \sum_{k \neq l} \rho_{kl},$$

  and no information on higher order correlation is used.

- Still, our criteria do not merely detect entanglement in the reduced two-qubit state!

Bound entanglement in spin chains

- Let us consider four spin-1/2 particles, interacting via the Hamiltonian

$$H = (h_{12} + h_{23} + h_{34} + h_{41}) + J_2(h_{13} + h_{24}),$$

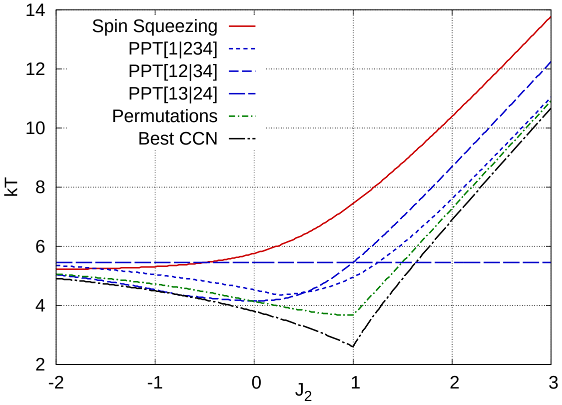
  where $h_{ij} = \sigma_x^{(i)} \otimes \sigma_x^{(j)} + \sigma_y^{(i)} \otimes \sigma_y^{(j)} + \sigma_z^{(i)} \otimes \sigma_z^{(j)}$ is a Heisenberg interaction between the qubits i, j.

- For the above Hamiltonian we compute the thermal state $\varrho(T, J_2) \propto \exp(-H/kT)$ and investigate its separability properties.

- For several separability criteria we calculate the maximal temperature, below which the criteria detect the states as entangled.

Bound entanglement in spin systems II

- Bound temperatures for entanglement



For $J_2 \gtrsim -0.5$, the spin squeezing inequality is the strongest criterion for separability. It allows to detect entanglement even if the state has a positive partial transpose (PPT) with respect to all bipartition.

Bound entanglement in spin systems III

- We found bound entanglement that is PPT with respect to all bipartitions in XY and Heisenberg chains, and also in XY and Heisenberg models on a completely connected graph, up to 10 qubits.
- Thus for these models, which appear in nature, there is a considerable temperature range in which the system is already PPT but not yet separable.

Bound entanglement in spin systems IV

- Simple example: Heisenberg system on a fully connected graph

$$H = J_x^2 + J_y^2 + J_z^2 = \frac{3N}{4} + \frac{1}{4} \sum_{k \neq l} \sigma_x^{(k)} \sigma_x^{(l)} + \sigma_y^{(k)} \sigma_y^{(l)} + \sigma_z^{(k)} \sigma_z^{(l)}.$$

- The ground state is very mixed: For large temperature range it is PPT bound entangled.
- The thermodynamics of this system can be computed analytically. Optimal spin squeezing inequalities are violated for $T < N$. [GT, PRA 71, 010301(R) (2005).]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Conclusions

- We presented a family of entanglement criteria that are able to detect any entangled state that can be detected based on the first and second moments of collective angular momenta.
- We explicitly determined the set of points corresponding to separable states in the space of first and second order moments.
- We applied our findings to examples of spin models, showing the presence of bound entanglement in these models.
- Presentation based on:
  GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL **99**, 250405 (2007);
  Recent results: arXiv:0806.1048.

*** THANK YOU ***