Detecting Genuine Multipartite Entanglement with Two Local Measurements

Géza Tóth[1,*] and Otfried Gühne[2,3,†]

[1]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, D-85748 Garching, Germany*
[2]*Institut für Theoretische Physik, Universität Hannover, Appelstraße 2, D-30167 Hannover, Germany*
[3]*Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, A-6020 Innsbruck, Austria*

We present entanglement witness operators for detecting *genuine* multipartite entanglement. These witnesses are robust against noise and require only two *local* measurement settings when used in an experiment, independent of the number of qubits. This allows detection of entanglement for an increasing number of parties without a corresponding increase in effort. The witnesses presented detect states close to Greenberger-Horne-Zeilinger, cluster, and graph states. Connections to Bell inequalities are also discussed.

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Entanglement lies at the heart of quantum mechanics and plays an important role in quantum information theory [1]. While bipartite entanglement is well understood, multiparty entanglement is still under intensive research. It was soon realized that it is not an extension of the bipartite case and several new phenomena have arisen. For instance, for three qubits there are two different classes of true many-body entanglement [2]. Moreover, multiqubit states can contradict local realistic classical models in a new and stronger way [3]. These phenomena can be used to implement novel quantum information processing tasks such as error correction [4], fault-tolerant quantum computation, cryptographic protocols such as secret sharing [5], measurement-based quantum computation [6] and open-destination teleportation [7].

With the rapid development of quantum control it is now possible to study experimentally the entanglement of many qubits using photons [7–10], trapped ions [11], or cold atoms on an optical lattice [12]. In these experiments it is not sufficient to claim that "the state is entangled". A multiqubit experiment is meaningful and presents something qualitatively new only if provably more than two qubits are entangled. While a lot of thought has been given to detecting entanglement in general [13–16], detection of *genuine* multiqubit entanglement has only a limited literature [9,10,15,17]. Existing methods need an experimental time growing *exponentially* with the number of qubits, making multiqubit entanglement detection impossible even for modest size systems.

We will show it is still possible to decide whether a state is multiqubit entangled without the need for exponentially growing resources, using only *local* measurements. This is unexpected since the property to be detected is *nonlocal* over increasing number of qubits. Our method can readily be used in any future experiment preparing Greenberger-Horne-Zeilinger (GHZ) and cluster states [6,18]. They both play a central role in the quantum algorithms mentioned before. GHZ states, as maximally entangled multiqubit states, are intensively studied [13–15] and have been realized in numerous experiments [8,9,11]. Cluster states

can easily be created in a spin chain with Ising-type interaction [18] and have been realized in optical lattices of two-state atoms [12]. Remarkably, their entanglement is more persistent to noise than that of a GHZ state [18].

A usual approach for detecting entanglement is using Bell inequalities [13–15]. These indicate the violation of local realism, a notion independent of quantum physics. When applied to detect quantum entanglement, they detect usually any (i.e., also partial or biseparable [19]) entanglement [15]. For N qubits Bell inequalities typically need the measurement of two variables at each qubit. Thus, as shown in Fig. 1(a), the number of local measurement settings needed increases exponentially with N. Here, a measurement setting means a simultaneous measurement of single qubit operators $\{O^{(k)}\}_{k=1}^{N}$ at sites $k = 1, 2, ..., N$ *in parallel*.

Another approach for detecting multipartite entanglement is using entanglement witnesses [16]. These are observables which have a positive or zero expectation value for all separable states, thus a negative expectation value signals the presence of entanglement. In a typical experiment one aims to prepare a pure state, $|\Psi\rangle$, and would like to detect it as true multipartite entangled.
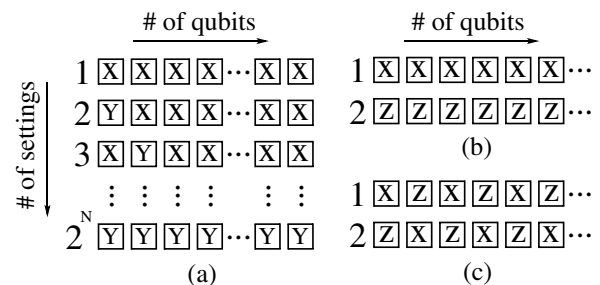

FIG. 1. (a) Measurement settings needed for detecting genuine multiqubit entanglement close to GHZ states with Bell inequalities. For each qubit the measured spin component is indicated. (b) Settings needed for the approach presented in this paper for detecting entangled states close to GHZ states and (c) cluster states.

While the preparation is never perfect, it is still expected that the prepared mixed state is in the proximity of $|\Psi\rangle$. The usual way to construct entanglement witnesses using the knowledge of this state is

$$\tilde{W} = \tilde{c}\mathbb{1} - |\Psi\rangle\langle\Psi|. \quad (1)$$

Here $\tilde{c}$ is the smallest constant such that for every product state $\mathrm{Tr}(\varrho\tilde{W}) \geq 0$. In order to measure the witness $\tilde{W}$ in an experiment, it must be decomposed into a sum of locally measurable operators [20]. The number of local measurements in these decompositions seems to increase exponentially with the number of qubits [10,17].

In this paper we propose to construct witnesses for N-qubit states of the form

$$\mathcal{W} = c_0\mathbb{1} - \sum_k c_k S_k, \quad (2)$$

where the c_k's are constants and the S_k operators stabilize the state $|\Psi\rangle$ [4]

$$S_k|\Psi\rangle = |\Psi\rangle. \quad (3)$$

For certain class of states, i.e., for GHZ and cluster states [18] the S_k's can be chosen locally measurable: they are the tensor products of Pauli spin matrices. It will turn out that for measuring our *stabilizer witnesses*, only two local measurement settings are required, independently of the number of qubits.

Let us briefly explain what we understand by such a local measurement setting [20]. Measuring a local setting $\{O^{(k)}\}_{k=1}^{N}$ consists of performing simultaneously the von Neumann measurements $O^{(k)}$ on the corresponding parties. After repeating the measurements several times, the coincidence probabilities for the outcomes are collected. Given these probabilities it is possible to compute all two-point correlations $\langle O^{(k)}O^{(l)}\rangle$, three-point correlations $\langle O^{(k)}O^{(l)}O^{(m)}\rangle$, etc. Since all these correlation terms can be measured with one setting, the number of settings determines the experimental effort rather than the number of measured correlation terms in Eq. (2). For detecting entanglement at least two settings are needed since the coincidence probabilities obtained from a single setting can always be mimicked by a separable state.

In order to demonstrate the power of our approach with an example, let us write down an entanglement witness (discussed later in detail) which detects *genuine* three-qubit entanglement around the three-qubit GHZ state $|\mathrm{GHZ}_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$:

$$\mathcal{W}_{\mathrm{GHZ}_3} := \frac{3}{2}\mathbb{1} - \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} - \frac{1}{2}[\sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(2)}\sigma_z^{(3)}$$
$$+ \sigma_z^{(1)}\sigma_z^{(3)}]. \quad (4)$$

This witness requires the measurement of the $\{\sigma_x^{(1)}, \sigma_x^{(2)}, \sigma_x^{(3)}\}$ and the $\{\sigma_z^{(1)}, \sigma_z^{(2)}, \sigma_z^{(3)}\}$ settings. The projector-based witness $\mathcal{W}_{\mathrm{GHZ}_3} = \mathbb{1}/2 - |\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3|$ requires four measurement settings [17].

After showing the previous example, we present a witness detecting entangled states close to an N-qubit GHZ state, $|\mathrm{GHZ}_N\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. Its stabilizing operators are

$$S_1^{(\mathrm{GHZ}_N)} := \prod_{k=1}^{N} \sigma_x^{(k)},$$
$$S_k^{(\mathrm{GHZ}_N)} := \sigma_z^{(k-1)}\sigma_z^{(k)} \quad \text{for } k = 2, 3, \ldots, N. \quad (5)$$

Using these stabilizing operators, Eq. (3) defines uniquely the GHZ state. The latter is stabilized not only by $S_k^{(\mathrm{GHZ}_N)}$'s but also by their products. These operators form a group called *stabilizer* [4], and $S_k^{(\mathrm{GHZ}_N)}$'s are the generators of this group. Allowing both $+1$ and -1 eigenvalues in Eq. (3), 2^N N-qubit states can be defined which are orthogonal to each other and form a complete basis. We will refer to this as the *GHZ state basis*. All the elements of the stabilizer are diagonal in this basis.

Theorem 1. The following entanglement witness detects genuine N-qubit entanglement for states close to an N-qubit GHZ state:

$$\mathcal{W}_{\mathrm{GHZ}_N} := 3\mathbb{1} - 2\left[\frac{S_1^{(\mathrm{GHZ}_N)} + \mathbb{1}}{2} + \prod_{k=2}^{N}\frac{S_k^{(\mathrm{GHZ}_N)} + \mathbb{1}}{2}\right]. \quad (6)$$

Another witness for this task is given by

$$\mathcal{W}'_{\mathrm{GHZ}_N} := (N-1)\mathbb{1} - \sum_{k=1}^{N} S_k^{(\mathrm{GHZ}_N)}. \quad (7)$$

Proof. First, we need to know that $\tilde{W}_{\mathrm{GHZ}_N} = \mathbb{1}/2 - |\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N|$ detects genuine N-qubit entanglement. This follows from the methods presented in Ref. [10]. We will now show that the witness $\tilde{W}_{\mathrm{GHZ}_N}$ is finer than the witness $\mathcal{W}_{\mathrm{GHZ}_N}$, i.e., that for all states with $\mathrm{Tr}(\varrho\mathcal{W}_{\mathrm{GHZ}_N}) < 0$ also $\mathrm{Tr}(\varrho\tilde{W}_{\mathrm{GHZ}_N}) < 0$ holds [21]. For that we have to show that $\mathcal{W}_{\mathrm{GHZ}_N} - \alpha\tilde{W}_{\mathrm{GHZ}_N} \geq 0$ where α is some positive constant. Then for any state ϱ detected by $\mathcal{W}_{\mathrm{GHZ}_N}$ we have $\alpha\mathrm{Tr}(\varrho\tilde{W}_{\mathrm{GHZ}_N}) \leq \mathrm{Tr}(\varrho\mathcal{W}_{\mathrm{GHZ}_N}) < 0$ thus the state is also detected by $\tilde{W}_{\mathrm{GHZ}_N}$. This implies that $\mathcal{W}_{\mathrm{GHZ}_N}$ is also a multiqubit witness. Let us now look at the observable $X := \mathcal{W}_{\mathrm{GHZ}_N} - 2\tilde{W}_{\mathrm{GHZ}_N}$ and show that $X \geq 0$. We can express X in the GHZ state basis. Since $\mathcal{W}_{\mathrm{GHZ}_N}$ as well as $\tilde{W}_{\mathrm{GHZ}_N}$ are diagonal in this basis, X is also diagonal. By direct calculation it is straightforward to check that the entries on the diagonal are all non-negative, which proves our claim. For the other witness one can show similarly that $\mathcal{W}'_{\mathrm{GHZ}_N} - 2\tilde{W}_{\mathrm{GHZ}_N} \geq 0$. $\square$

The main advantage of the witnesses $\mathcal{W}_{\mathrm{GHZ}_N}$ and $\mathcal{W}'_{\mathrm{GHZ}_N}$ in comparison with $\tilde{W}_{\mathrm{GHZ}_N}$ lies in the fact that for implementing them *only two measurement settings* are needed as shown in Fig. 1(b). From the first setting $\langle S_1^{(\mathrm{GHZ}_N)}\rangle$ can be obtained, from the second one $\langle S_k^{(\mathrm{GHZ}_N)}\rangle$

for $k = 2, 3, \ldots, N$. The form of $\mathcal{W}_{\text{GHZ}_N}$ can be intuitively understood as follows. The first term in the square bracket is a projector to the subspace where $\langle S_1^{(\text{GHZ}_N)} \rangle = +1$. The second one is a projector to subspace where $\langle S_k^{(\text{GHZ}_N)} \rangle = +1$ for all $k \in \{2, 3, \ldots, N\}$. Clearly only a GHZ state gives $+1$ for both projectors. The witness $\mathcal{W}_{\text{GHZ}_N}$ can be proven to be optimal from the point of view of noise tolerance among stabilizer witnesses using two measurement settings, and having the property $\mathcal{W}_{\text{GHZ}_N} - 2\tilde{\mathcal{W}}_{\text{GHZ}_N} \geq 0$ [22].

For practical purposes it is important to know how large neighborhood of the GHZ state is detected by our witnesses. This is usually characterized by the robustness to noise. The witness $\mathcal{W}_{\text{GHZ}_N}$ is very robust: It detects a state of the form $\varrho(p) = p_{\text{noise}} \mathbb{1}/2^N + (1 - p_{\text{noise}})|\text{GHZ}_N\rangle \times \langle \text{GHZ}_N|$ for $p_{\text{noise}} < 1/(3 - 4/2^N)$ as true multipartite entangled thus it tolerates at least 33% noise, independent of the number of qubits. For $N = 3$ the witness from Eq. (6) was already given in Eq. (4) and tolerates noise up to $p_{\text{noise}} < 0.4$. The witness $\mathcal{W}'_{\text{GHZ}_N}$, having the minimal N stabilizing terms, is not so robust: It tolerates noise for $p_{\text{noise}} < 1/N$.

Other novel witnesses can be obtained by including further terms of the stabilizer and using more than two measurement settings. For instance, following the lines of the previous paragraphs it can be proved that the observable $\mathcal{W}''_{\text{GHZ}_3} := 2\mathbb{1} - S_1^{(\text{GHZ}_3)}[\mathbb{1} + S_2^{(\text{GHZ}_3)}][\mathbb{1} + S_3^{(\text{GHZ}_3)}] = 2\mathbb{1} + \sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)} + \sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)} + \sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)} - \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}$ detects genuine three-party entanglement if $p_{\text{noise}} < 1/2$. It is very remarkable that witness $\mathcal{W}''_{\text{GHZ}_3}$ is equivalent to Mermin's inequality [14] for detecting violation of local realism. However, Mermin's inequality in the form from above is normally used to detect some, not necessarily genuine multipartite, entanglement. From our witness it follows that it detects indeed only genuine multipartite entanglement [23]. For $N > 3$ Mermin's inequality also contains only stabilizing terms. Including even more terms from the stabilizer one can even construct the projector-based witness [24].

Let us continue our discussion by presenting a witness detecting entangled states close to cluster states. An N-qubit cluster state, $|C_N\rangle$, can be created starting from the state $|1111\ldots\rangle_x$ by applying the Ising chain-type dynamics $U_{cl} = \exp[i\frac{\pi}{4}\sum_k (1 - \sigma_z^{(k)})(1 - \sigma_z^{(k+1)})]$. The stabilizing operators used for constructing our witnesses are

$$S_1^{(C_N)} := \sigma_x^{(1)}\sigma_z^{(2)},$$
$$S_k^{(C_N)} := \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)} \quad \text{for } k = 2, 3, \ldots, N-1, \quad (8)$$
$$S_N^{(C_N)} := \sigma_z^{(N-1)}\sigma_x^{(N)}.$$

The results for cluster states are analogous to the case of the GHZ state.

Theorem 2. The following witnesses detect genuine N-party entanglement close to a cluster state

$$\mathcal{W}_{C_N} := 3\mathbb{1} - 2\left[\prod_{\text{even } k} \frac{S_k^{(C_N)} + \mathbb{1}}{2} + \prod_{\text{odd } k} \frac{S_k^{(C_N)} + \mathbb{1}}{2}\right],$$

$$\mathcal{W}'_{C_N} := (N-1)\mathbb{1} - \sum_{k=1}^{N} S_k^{(C_N)}. \quad (9)$$

Proof. In order to show that these observables are witnesses, we first show that

$$\tilde{\mathcal{W}}_{C_N} := \frac{1}{2}\mathbb{1} - |C_N\rangle\langle C_N| \quad (10)$$

is a witness. To do this we have to show that for all pure biseparable states $|\phi\rangle$ the bound $|\langle\phi|C_N\rangle| \leq 1/\sqrt{2}$ holds. This is equivalent to showing that the Schmidt coefficients do not exceed $1/\sqrt{2}$ when making a Schmidt decomposition of $|C_N\rangle$ with respect to an arbitrary bipartite splitting, since they bound the overlap with the biseparable states [10]. It is known that one can produce a singlet between an arbitrary pair of qubits from a cluster state by local operations and classical communication [18]. For a singlet both Schmidt coefficients are $1/\sqrt{2}$. Furthermore, it is known that the largest Schmidt coefficient cannot decrease [25] under these operations. This proves our claim. Knowing that $\tilde{\mathcal{W}}_{C_N}$ is a witness, one can show as in the GHZ case that $\mathcal{W}_{C_N}$ and $\mathcal{W}'_{C_N}$ are also witnesses. $\square$

The stabilizing operators in the expression given for $\mathcal{W}_{C_N}$ are again grouped into two terms corresponding to the two settings shown in Fig. 1(c). The witness $\mathcal{W}_{C_N}$ tolerates mixing with noise if $p_{\text{noise}} < 1/(4 - 4/2^{N/2})$ for even N (respectively, $p_{\text{noise}} < 1/[4 - 2(1/2^{(N+1)/2} + 1/2^{(N-1)/2})]$ for odd N). Thus, for any number of qubits at least 25% noise is tolerated. Alternatively, $\tilde{\mathcal{W}}_{C_N}$ can also be decomposed into local terms following Refs. [10,17]. The noise tolerance is at least 50% even for large N, however, more than the two settings are necessary.

Up to now, we presented witnesses detecting *only* genuine N-qubit entanglement. If the noise is large, there might be no true N-party entanglement in the system. In this case *some* entanglement can still be detected with the two measurement settings from above, although it may not be multipartite entanglement. Similarly to Ref. [26], the following necessary conditions for full separability can be constructed for GHZ and cluster states

$$\langle S_1^{(\text{GHZ}_N)} \rangle + \langle S_m^{(\text{GHZ}_N)} \rangle \leq 1 \quad \text{for } N \geq m \geq 2, \quad (11)$$

$$\langle S_k^{(C_N)} \rangle + \langle S_{k+1}^{(C_N)} \rangle \leq 1 \quad \text{for } N-1 \geq k \geq 1. \quad (12)$$

These conditions detect entanglement after mixing with noise if $p_{\text{noise}} < 1/2$ and they both need only two measurement settings. The proofs are given in the Appendix.

The previous results can straightforwardly be generalized for graph states [27]. These states are defined by a graph of N vertices. Edges of this graph are described by the adjacency matrix Γ. $\Gamma_{kl} = 1$ (0) if the vertices k and l are connected (not connected). An N-qubit state is defined as an eigenstate with eigenvalue 1 of the stabilizing operators $S_k^{(G_N)} := \sigma_x^{(k)} \prod_{l \neq k} (\sigma_z^{(l)})^{\Gamma_{kl}}$. Physically, $\Gamma_{kl} = 1$ (0) means that spins k and l interact (do not interact) by an Ising-type interaction. We assume that the graph cannot be partitioned into two separate subgraphs, since then the graph state would be biseparable.

A witness detecting genuine N-party entanglement can be defined as $\mathcal{W}_{G_N} := (N-1)\mathbb{1} - \sum_k S_k^{(G_N)}$. The proof is essentially the same as before. It must be used that one can produce from a graph state by local means a singlet between an arbitrary pair of qubits [28]. For two-colorable graphs only two settings are needed for measuring $\mathcal{W}_{G_N}$ [29]. The maximum number of settings required is N, reached, for example, by the state corresponding to the complete graph. A necessary condition for separability can be given as $\langle S_k^{(G_N)} \rangle + \langle S_m^{(G_N)} \rangle \leq 1$ where spins (k) and (m) are neighbors.

In summary, based on the stabilizer theory we constructed entanglement witnesses with simple local decomposition for GHZ, cluster, and graph states. Our approach is optimal from the point of view of the duration of an experimental implementation since only two local measurement settings are needed independent of the number of qubits. We found that some Bell inequalities (when used for entanglement detection) and the projector-based witnesses are in fact also stabilizer witnesses.

Appendix: Proof of Equations (11) and (12). Using the Cauchy-Schwarz inequality and the fact that $\langle \sigma_x^{(k)} \rangle^2 + \langle \sigma_z^{(k)} \rangle^2 \leq 1$ we obtain for product states $\langle S_1^{(\mathrm{GHZ}_N)} \rangle + \langle S_m^{(\mathrm{GHZ}_N)} \rangle \leq |\langle \sigma_x^{(m-1)} \rangle||\langle \sigma_x^{(m)} \rangle| + |\langle \sigma_z^{(m-1)} \rangle||\langle \sigma_z^{(m)} \rangle| \leq 1$ for $m = 2, 3, \ldots, N$. Because of linearity, this bound is also valid for full separable states. For the second inequality, we have $\langle S_k^{(C_N)} + S_{k+1}^{(C_N)} \rangle = \langle \sigma_z^{(k-1)} \rangle \langle \sigma_x^{(k)} \rangle \langle \sigma_z^{(k+1)} \rangle + \langle \sigma_z^{(k)} \rangle \langle \sigma_x^{(k+1)} \rangle \langle \sigma_z^{(k+2)} \rangle \leq |\langle \sigma_x^{(k)} \rangle| \times |\langle \sigma_z^{(k+1)} \rangle| + |\langle \sigma_z^{(k)} \rangle||\langle \sigma_x^{(k+1)} \rangle| \leq 1$. Here, for the end of the chain $\sigma_z^{(0)} = \sigma_z^{(N+1)} = \mathbb{1}$ was used. $\qquad\square$

*Electronic address: toth@alumni.nd.edu

†Electronic address: otfried.guehne@uibk.ac.at

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