Rapid Communications

Entanglement and extreme planar spin squeezing

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We introduce an entanglement-depth criterion optimized for planar quantum-squeezed (PQS) states. It is connected with the sensitivity of such states for estimating a phase generated by rotations about an axis orthogonal to its polarization. We compare numerically our criterion with the well-known extreme spin-squeezing condition of Sørensen and Mølmer [Phys. Rev. Lett. **86**, 4431 (2001)] and show that our condition detects a higher depth of entanglement when both planar spin variances are squeezed below the standard quantum limit. We employ our theory to monitor the entanglement dynamics in a PQS state produced via quantum nondemolition measurements using data from a recent experiment [Phys. Rev. Lett. **118**, 233603 (2017)].

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Introduction. Detecting entanglement in large quantum systems is a major goal in quantum information science and underpins the development of quantum technologies [1,2]. Attention has now shifted toward the practical use of entanglement as a resource: In particular, entanglement-enhanced sensing using ensembles of 10^3–10^{12} atomic spins has emerged as a major application [3,4]. In this context, spin-squeezing inequalities can be used to quantify entanglement-enhanced sensitivity. Standard treatment studies spin-squeezed states (SSSs), characterized by a large spin polarization in the y direction and a small variance in the z direction via the parameter $\xi_s^2 := \frac{N(\Delta J_z)^2}{|\langle J_y\rangle|^2}$, where $J_v = \sum_{n=1}^N j_v^{(n)}$ for $v = x,y,z$ are the collective spin components, $j_v^{(n)}$ are single-particle spin operators, and N is the total number of atoms. Then, states with $\xi_s^2 < 1$ provide quantum-enhanced sensitivity for estimating phases $\phi \approx 0$ due to small rotations around J_x [5,6]. Such states have been produced using various platforms, including cold atoms [7–18], trapped ions [19], magnetic systems [20], and photons [21].

Their metrological sensitivity is strongly connected to entanglement: $\xi_s^2 < 1$ also implies entanglement for atoms with spin $j = 1/2$ [22]. More quantitatively, the amount of spin squeezing is also connected with the so-called *depth of entanglement*, i.e., the number of particles in the largest separable subset [23]. Several other highly entangled states have recently been found useful for quantum metrology. For example, Dicke states, which are unpolarized and have a large value of $\langle J_x^2 + J_y^2\rangle$ and a small variance in the z direction. Spin-squeezing inequalities have been developed to characterize entanglement in such states [24,25], which have been produced in experiments with photons [26,27] and Bose-Einstein condensates [24,28–31].

Here, we focus on so-called *planar quantum-squeezed* (PQS) states, studied theoretically in Refs. [32–34], and produced in a recent experiment [35,36]. They have reduced spin variances in two directions, i.e., $(\Delta J_\parallel)^2 := (\Delta J_y)^2 + (\Delta J_z)^2$ is small, and a large in-plane polarization, i.e., $\langle J_y\rangle \approx Nj$. They provide quantum-enhanced sensitivity in estimating phases generated by rotations about the $\hat{x}$ axis (see Fig. 1 for an illustration) and are useful for tracking a changing phase shift or simultaneous estimation of phase and amplitude beyond classical limits [35,36]. The planar squeezing parameter,

$$\xi_\parallel^2 := \frac{(\Delta J_\parallel)^2}{|\langle J_\parallel\rangle|}, \tag{1}$$

where $|\langle J_\parallel\rangle| := \sqrt{\langle J_y\rangle^2 + \langle J_z\rangle^2}$ is the in-plane polarization, was introduced by He and co-workers [32,33] to quantify such enhanced sensitivity and detect entanglement. The relation between their metrological usefulness and their degree of multiparticle entanglement is explored here.

In this Rapid Communication, we introduce a method to detect the depth of entanglement based on the planar-squeezing parameter $\xi_\parallel^2$. We present the condition,

$$\xi_\parallel^2 \geqslant \zeta_J^2, \tag{2}$$

where ζ_J^2 is the minimum value of the planar-squeezing parameter *over single-particle states* of spin J. We prove that, for all spin-j ensembles that contain groups of at most k-entangled particles, called k producible [23,37], Eq. (2) holds with $J = kj$. Thus, $\xi_\parallel^2 < \zeta_J^2$ implies a depth of entanglement of at least $k+1 = J/j + 1$. We can even estimate at least how many particles must be in fully entangled $(k+1)$-particle groups. We stress that our criterion is very simple to use. We need to calculate ζ_J^2 only once for the relevant range of J, then

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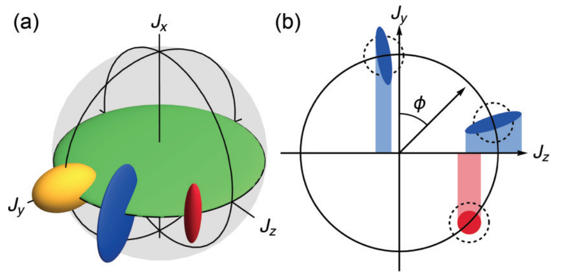

FIG. 1. (a) Bloch-sphere representation of: (yellow horizontal spheroid) spin-coherent states, including uncertainty in J_y arising from Poissonian fluctuations in the preparation; (blue vertical disk) a SSS produced by squeezing the J_z variance; (red vertical spheroid) a PQS state produced by squeezing both J_z and J_y; (green horizontal disk) a Dicke state. (b) Sensitivity advantage of a PQS state compared to a SSS in detecting an unknown phase ϕ. The dashed black circles indicate the shot-noise limit $\Delta\phi = 1/\sqrt{N}$. The SSS provides enhanced sensitivity for detecting phases around $\phi \simeq 0$ but reduced sensitivity for phases around $\phi \simeq \pm\pi/2$. In contrast, although the sensitivity of the PQS state is slightly worse around $\phi \simeq 0$, it provides enhanced sensitivity for all phases $0 \leqslant \phi \leqslant 2\pi$.

Eq. (2) can be applied for entanglement detection without any additional numerical optimization.

Finally, we examine the usefulness of our criterion. We compare it to the well-known criterion introduced by Sørensen and Mølmer in Ref. [23] and find that ours detects a higher entanglement depth even for nonideal PQS. We also test our theory using data from a recent experiment in which a PQS state was generated via a semicontinuous quantum nondemolition (QND) measurement [35,36].

Link between our parameter and metrological sensitivity. We consider a protocol in which a collective spin state is rotated about J_x and accumulates a phase ϕ such that $J_z^{\text{out}} = J_z^{\text{in}} \cos\phi - J_y^{\text{in}} \sin\phi$. Afterwards, the phase is inferred from repeated measurements of J_z^{out} with a sensitivity given by the error-propagation formula $(\Delta\phi)^2 = (\Delta J_z^{\text{out}})^2/|\partial_\phi \langle J_z^{\text{out}}\rangle|^2$. We consider as reference an input state with uncertainties at the standard quantum limit (SQL) $(\Delta J_y^{\text{in}})^2_{\text{SQL}} = (\Delta J_z^{\text{in}})^2_{\text{SQL}} = \frac{1}{2}|\langle J_\parallel^{\text{in}}\rangle|$ [34,36]. A SQL-limited state cannot beat the shot-noise limit corresponding to separable states since $(\Delta\phi)^2_{\text{SQL}} = |\langle J_\parallel^{\text{in}}\rangle|/(\langle J_z^{\text{in}}\rangle^2 \cos^2\phi + \langle J_y^{\text{in}}\rangle^2 \sin^2\phi) \geqslant 1/N$. Hence, we normalize the sensitivity with respect to the SQL and obtain $(\Delta\phi)^2/(\Delta\phi)^2_{\text{SQL}} = [(\Delta J_z^{\text{in}})^2 \cos^2\phi + (\Delta J_y^{\text{in}})^2 \sin^2\phi]/|\langle J_\parallel^{\text{in}}\rangle|$. By averaging over ϕ we find

$$\int_0^{2\pi} \frac{d\phi}{2\pi} (\Delta\phi)^2/(\Delta\phi)^2_{\text{SQL}} = \frac{1}{2}\xi_\parallel^2. \tag{3}$$

Thus, the parameter appearing on the left-hand side of Eq. (2) also quantifies the average sensitivity enhancement over the interval $0 \leqslant \phi \leqslant 2\pi$ compared to the SQL.

Entanglement criterion for planar squeezing. Following an approach similar to past works [23,25], we derive a tight criterion to detect the depth of entanglement by computing the function,

$$G_k^{(j)}(X) := \frac{1}{kj} \min_{\substack{\phi\in(\mathbb{C}^d)^{\otimes k} \\ \frac{1}{kj}\langle L_y\rangle_\phi = X}} \left[(\Delta L_y)^2_\phi + (\Delta L_z)^2_\phi\right], \tag{4}$$

where $d = 2j+1$, j is the single-particle spin quantum number, and L_v's are *collective k-particle spin operators*, i.e., $L_v = \sum_{n=1}^k j_v^{(n)}$, where $j_v^{(n)}$'s are single-particle spin-j components. First, we find a tight lower bound on the planar spin variance valid for all states with a depth of entanglement smaller than k.

Observation 1. Every k-producible state of a spin-j particle system with an average number of particles $\langle N\rangle$ must satisfy the tight inequality,

$$(\Delta J_\parallel)^2 \geqslant \langle N\rangle j \mathcal{G}_k^{(j)}\left(\tfrac{|\langle J_\parallel\rangle|}{\langle N\rangle j}\right), \tag{5}$$

where $\mathcal{G}_k^{(j)}$ is defined as the convex hull of (4). Thus, every state that violates Eq. (5) must have a depth of entanglement of at least $k+1$.

Proof. For pure k-producible states of (constant) N particles we have $(\Delta J_\parallel)^2_N = \sum_n [(\Delta L_y^{(n)})^2 + (\Delta L_z^{(n)})^2] \geqslant \sum_n k_n j \mathcal{G}_{k_n}^{(j)}(\langle L_y^{(n)}\rangle/k_n j)$, where $L_v^{(n)}$'s are collective operators of $0 \leqslant k_n \leqslant k$ particles. The second inequality follows directly from the definition of $\mathcal{G}_{k_n}^{(j)}$. Now, we use that $\mathcal{G}_{k_n}^{(j)}$'s are as follows: (i) convex and (ii) decreasing for increasing the index, i.e., $\mathcal{G}_r^{(j)} \leqslant \mathcal{G}_s^{(j)}$ for $r \geqslant s$ and that $k \geqslant k_n$ and $\sum_n k_n = N$. Then, $\sum_n k_n j \mathcal{G}_{k_n}^{(j)}(\langle L_y^{(n)}\rangle/k_n j) \geqslant \sum_n k_n j \mathcal{G}_k^{(j)}(\langle L_y^{(n)}\rangle/k_n j) \geqslant N j \mathcal{G}_k^{(j)}(\langle J_y\rangle/Nj)$ follows where the first inequality comes from property (ii) and the second comes from property (i) and Jensen's inequality. Clearly, if N is divisible by k, then the inequality (5) is tight by construction. Let us consider now a state with a nonzero particle number variance $\varrho = \sum_N Q_N \varrho_N$, where ϱ_N's are states with fixed N's and Q_N's are probabilities. From the properties above it follows that $(\Delta J_\parallel)^2 \geqslant \sum_N Q_N (\Delta J_\parallel)^2_N \geqslant \sum_N Q_N N j \mathcal{G}_k^{(j)}(\langle J_y\rangle/Nj) \geqslant \langle N\rangle j \mathcal{G}_k^{(j)}(\langle J_y\rangle/\langle N\rangle j)$ holds, $\langle N\rangle = \sum_N Q_N N$ being the average particle number. ■

Numerical computation of $\mathcal{G}_k^{(j)}$. In order to detect the depth of entanglement with our criterion we need to carry out the optimization in Eq. (4) and then construct the convex hull $\mathcal{G}_k^{(j)}$ that has properties (i) and (ii) mentioned in the proof of Observation 1. For $k=1$ and $j=1$, straightforward algebra yields $G_1^{(1)}(X) = \frac{3}{2} - X^2 - \frac{1}{2}\sqrt{1-X^2}$. Analytical expressions are very hard to obtain even for the next simplest cases.

Numerically, the problem of finding the convex hull $\mathcal{G}_k^{(j)}$ can be approached exploiting the Legendre transform in this framework defined as [38,39]

$$\mathcal{L}[(\Delta L_\parallel)^2_\phi/kj](T) := \inf_\phi \left[\frac{1}{kj}(\Delta L_\parallel)^2_\phi - \langle T\rangle_\phi\right] \tag{6}$$

for the normalized planar variance $(\Delta L_\parallel)^2_\phi/kj$ as a function of $T = L_y/kj$. Then, the lower bound $(\Delta L_\parallel)^2_\phi \geqslant \mathcal{G}_k^{(j)}(X)$ is obtained by means of another Legendre transform,

$$\mathcal{G}_k^{(j)}(X) := \sup_\lambda \left\{\lambda X - \mathcal{L}[(\Delta L_\parallel)^2_\phi/kj](\lambda L_y/kj)\right\}, \tag{7}$$

where X is a real number. The function (7) is precisely the convex hull that we are looking for. Furthermore, (6) can be written as an eigenvalue problem (see also Refs. [40–42]

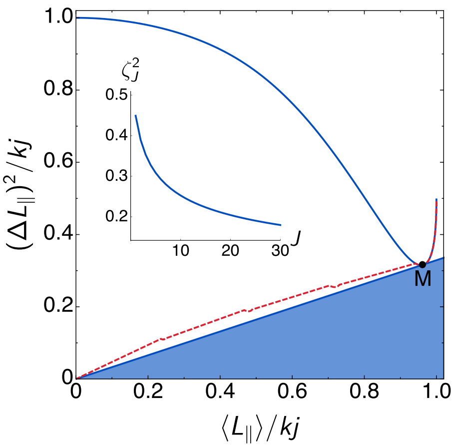

FIG. 2. Lower bounds to $(\Delta L_\parallel)^2/kj$ as functions of $\langle L_\parallel\rangle/kj$ for a system of k spin-j particles. The cases of $k=4$ and $j=1$ are shown. (The dashed curve) The function $G_4^{(1)}(X)$. (The solid line) The function $G_4^{\rm sy}(X)$ computed on the symmetric subspace. The convex hull of $G_4^{(1)}(X)$, denoted by $\mathcal{G}_4^{(1)}(X)$ is a linear function for $X \leqslant X_{\min} = \mathrm{argmin}[G_4^{\rm sy}(X)/X]$, whereas for $X > X_{\min}$ it coincides with $G_4^{\rm sy}(X)$. M denotes the point of the curve for which $X = X_{\min}$. The straight line provides a lower bound on $\mathcal{G}_4^{(1)}(X)$. (The inset) The parameter ζ_J^2 as a function of J.

addressing similar problems),

$$\mathcal{L}\big[(\Delta L_\parallel)_\phi^2/kj\big](\lambda L_y/kj) = \frac{1}{kj}\min_{s_y,s_z}\big[\min_\phi \langle H_{s_y,s_z,\lambda}\rangle_\phi\big], \quad (8)$$

where the Hamiltonian $H_{s_y,s_z,\lambda} = (L_y - s_y)^2 + (L_z - s_z)^2 - \lambda L_y$ is a collective operator acting on a k-partite space of spin-j particles. Moreover, by writing a general pure state (here we consider integer values of kj) as $|\phi\rangle = \sum_{J=0}^{kj} a_J|\psi_J\rangle$, i.e., as a superposition of single spin-J states $|\psi_J\rangle$, the expectation value in Eq. (8) can be written as

$$\langle H_{s_y,s_z,\lambda}\rangle_\phi = \sum_{J=0}^{kj} a_J^2 \big\langle \big(L_y^{(J)} - s_y\big)^2 + \big(L_z^{(J)} - s_z\big)^2 - \lambda L_y^{(J)}\big\rangle_{\psi_J}, \quad (9)$$

where $L_m^{(J)}$'s are single spin-J operators. In particular, for $k>1$ and the kj integer we can easily prove that $\mathcal{G}_k^{(j)}(0) = 0$ where the value on the right-hand side is reached for $|\phi\rangle$ being the singlet. More in general, substituting Eqs. (8) and (9) into Eq. (7) one can see that the function $\mathcal{G}_k^{(j)}(X)$ can be obtained with minimizations in spin-J subspaces with $0 \leqslant J \leqslant kj$ ($1/2 \leqslant J \leqslant kj$ for the kj half-integer). Thus, by increasing k one has to minimize over a larger number of subspaces and consider a higher number of parameters a_J, which makes the resulting function decreasing with k, which is just property (ii) needed in the proof of Observation 1. When the minimization problem (4) is restricted to the symmetric subspace $J = kj$ then we call the resulting function $G_J^{\rm sy}(X)$. Its convex roof can be obtained based on Eq. (9) if we set $a_{kj} = 1$. In Fig. 2, we present a concrete example. The function $G_4^{(1)}(X)$ is plotted together with its convex hulls $\mathcal{G}_4^{(1)}(X)$ and $G_4^{\rm sy}(X)$. We see in the figure that a simple linear function can be used as a lower bound to $\mathcal{G}_4^{(1)}(X)$. This lower bound works in general, as we show in what follows.

TABLE I. Values of ζ_J^2 for $0 \leqslant J \leqslant 27$.

J	ζ_J^2	J	ζ_J^2	J	ζ_J^2
1	0.45	10	0.26067	19	0.21111
2	0.44906	11	0.25262	20	0.20758
3	0.38945	12	0.2455	21	0.20428
4	0.35321	13	0.23913	22	0.20118
5	0.32779	14	0.23338	23	0.19826
6	0.30852	15	0.22815	24	0.19551
7	0.29318	16	0.22336	25	0.1929
8	0.28054	17	0.21896	26	0.19043
9	0.26986	18	0.21489	27	0.18809

Linear lower bound. As outlined above, the computation of $\mathcal{G}_k^{(j)}(X)$ still requires some numerics, which can be hard for high k and j. Here, we simplify further this task by finding a suitable lower bound that requires only the numerical computation of $G_J^{\rm sy}(X)$ with $J = kj$ and is thus easier than computing the full $\mathcal{G}_k^{(j)}(X)$.

Observation 2. A convex lower bound to the curve $G_k^{(j)}(X)$ defined as in Eq. (4) is given by

$$\mathcal{G}_k^{(j)}(X) \geqslant X\zeta_J^2, \quad (10)$$

where $\zeta_J^2 := \min_{|\psi_k\rangle}[(\Delta L_y)_{\psi_k}^2 + (\Delta L_z)_{\psi_k}^2]/\langle L_y\rangle_{\psi_k}$ is the minimum value of the planar squeezing parameter over single-particle states $|\psi_k\rangle$ of spin $J = kj$. The proof is given in the Appendix.

With this method we need only to determine ζ_J^2 for the relevant range of J, which can be written as $\zeta_J^2 = \min_X[G_J^{\rm sy}(X)/X]$. Thus, as a simple algorithm one can: (i) Find the ground states $|\phi_\lambda\rangle$ of H_λ *restricted to the symmetric subspace*; (ii) compute $(\Delta L_\parallel)_{\phi_\lambda}^2$ and $\langle L_y\rangle_{\phi_\lambda}$; and finally take $\zeta_J^2 = \min_{\phi_\lambda}(\Delta L_\parallel)_{\phi_\lambda}^2/\langle L_y\rangle_{\phi_\lambda}$, which is feasible until very large J, up to the thousands. As an example the values of ζ_J^2 up to $J = 27$ are given in Table I, whereas the qualitative behavior can be observed in the inset of Fig. 2. Note that Eq. (10) is a tight approximation only for $k \geqslant 2$, independent of j. For $k=1$ the original criterion given in Eq. (5) has to be used instead.

From Observations 1 and 2, we immediately obtain Eq. (2), which connects the metrological performance of PQS states to their entanglement depth. Next, we show that, apart from proving that the entanglement depth is $k+1$, we also obtain information about how many particles are in fully entangled groups of $(k+1)$. This provides a simple interpretation of the degree of the violation of Eq. (2).

Observation 3. Let us assume that the total polarization is equally distributed over all particles. Then, there is at least a fraction $f_{k+1} = (1 - \xi_\parallel^2/\zeta_J^2)$ of particles in fully entangled groups of $(k+1)$ or more with k given by J/j. The proof is given in the Appendix where the case of varying particle numbers is included in the model. We discuss that, without the assumption of equally split polarization, the above statement still holds for almost totally polarized states, i.e., with $\langle J_y\rangle \approx Nj$ and that similar ideas work also for the Sørensen-Mølmer criterion.

Practical use of the criterion. Thus, our criterion can be employed to detect the depth of entanglement whenever two

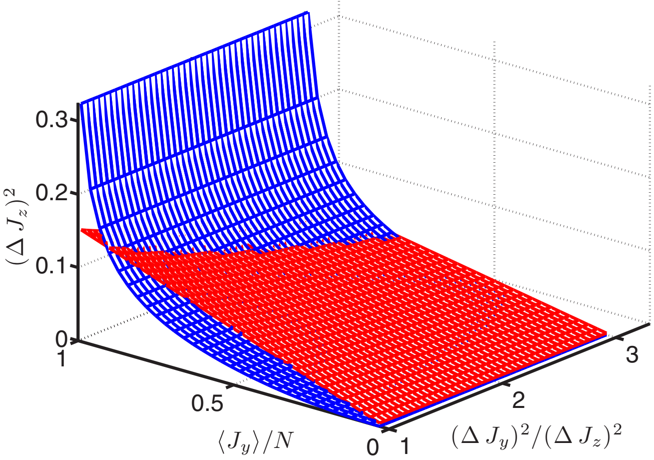

FIG. 3. Lower bound on $(\Delta J_z)^2$ given by the (blue) extreme spin-squeezing and (red) planar-squeezing criteria taken for $k = 5$ and $j = 1$ as a function of the ratio between the two planar spin variances and the in-plane polarization. Our criterion detects a depth of entanglement higher than that of Sørensen-Mølmer for the parameter values for which the red plot is above the blue one.

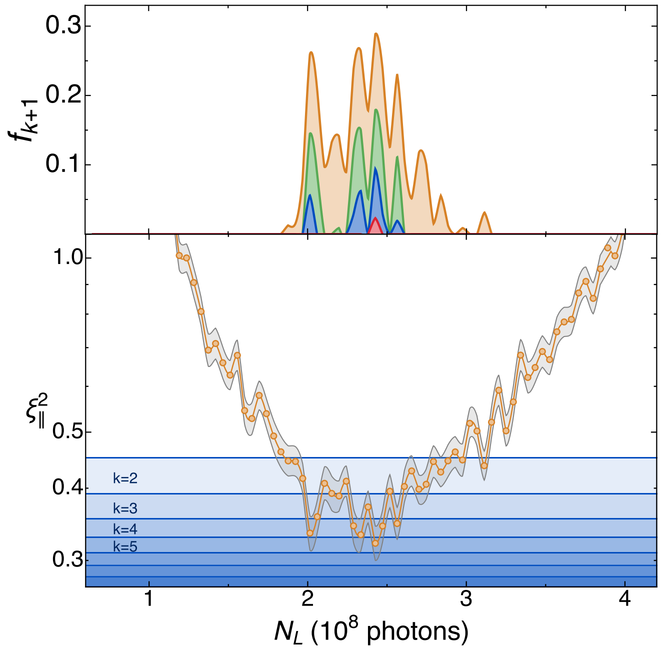

FIG. 4. Top: The shaded curves: lower bound for the number of atoms in fully entangled groups of at least $(k+1)$ particles. From top to bottom: $k = 1$–4. Bottom: The squeezing parameter $\xi_\parallel^2$ as a function of the number of photons N_L used in the QND measurement. The orange shaded area represents the $\pm 1\sigma$ confidence interval.

collective spin variances are known as well as the total in-plane polarization. With the same input information, it would be possible to use also the Sørensen-Mølmer extreme spin-squeezing condition [43]. Then, we can numerically compare the two criteria and study in which cases our criterion is more suitable to detect entanglement. To do this we parametrize the states with the ratio $\alpha = (\Delta J_z)^2/(\Delta J_y)^2$ between the two spin variances and the total in-plane polarization $\beta = \langle J_y \rangle / N$. We plot the lower bound on $(\Delta J_z)^2$ for $k = 5$ and $j = 1$ for various values of α and β and see for which regions of the (α,β) plane our criterion detects a higher depth. The result is shown in Fig. 3 where we can observe that our criterion detects a higher degree of entanglement on most of the plane, especially whenever the two variances become equal. Vice versa, totally polarized states that are spin squeezed only along J_z are detected with a higher depth by the criterion of Sørensen-Mølmer. All these statements are valid also for other k and j values. Thus, our criterion is especially tailored for detecting PQS states and distinguishing those from traditional spin-squeezed states, which are optimally detected by the criterion of Sørensen-Mølmer. Furthermore, the linearity of our criterion makes it directly connected with improved sensitivity in phase estimations: A value of $\xi_\parallel^2$ below the threshold given by ζ_J^2 with $J = kj$ implies that: (1) the state must be $(k+1)$ entangled and (2) its average sensitivity to rotations about the axis orthogonal to the plane of squeezing as compared to the SQL is better than that of any state with depth of entanglement k or lower.

Next, we employ our criterion (2) to analyze entanglement in a PQS state produced in a recent experiment with an ensemble of $N = 1.75 \times 10^6$ cold ^{87}Rb atoms via semicontinuous QND measurements [36]. In Fig. 4 we plot the observed planar-squeezing parameter $\xi_\parallel^2$ as a function of the measurement strength, parametrized by the number of photons N_L used in the QND measurement. As N_L increases, the input spin-coherent state evolves into a planar-squeezed state with squeezing observed between $N_L \simeq 2 \times 10^8$ and $N_L \simeq 3 \times 10^8$ photons after which the spin variances increase due to noise and decoherence introduced by off-resonant scattering of probe photons. We also plot the corresponding fraction f_{k+1} of atoms in fully entangled groups of $(k+1)$ or more, detected using our criterion. We observe the corresponding increase in entanglement depth with N_L up to the optimum of $N_L = 2.47 \times 10^8$ photons after which entanglement is gradually lost. At the optimum N_L we observe a spin-coherence $\langle J_\parallel \rangle = 0.83N$ and a planar-squeezing parameter $\xi_\parallel^2 = 0.32 \pm 0.02$. For comparison, using the criterion developed by He and co-workers [32,33], one would detect a fraction 0.39 of atoms in entangled states without any information about the depth of entanglement. The details of the experiment are given in the Supplemental Material [44] (see also Ref. [45]).

Conclusions. We have introduced a criterion suitable to detect the depth of entanglement in planar-squeezed states and to distinguish them from traditional spin-squeezed states, detectable with the criterion of Sørensen-Mølmer [23]. Our criterion is simple to evaluate and directly connected with the sensitivity of the PQS states for phase estimations that do not require any prior knowledge of the phase. By numerical comparison, we have also shown that our criterion represents an important alternative to that of Sørensen-Mølmer suitable to detect entanglement in PQS states. Finally, we tested our criterion with data from a recent experiment in which a PQS state was generated via semicontinuous QND measurement [36].

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Appendix. Proof of Observation 2. Let us consider a general pure k-particle state $|\phi\rangle = \sum_J a_J|\psi_J\rangle$ where each $|\psi_J\rangle$ is a single spin-J particle state. We assume that the mean planar spin points into the y direction. We now need that the collective k-particle spin components L_v can be written as the direct sum of operators $L_v^{(J)}$ acting on spin-J particle spaces with $0 \leqslant J \leqslant kj$ (or $1/2 \leqslant J \leqslant kj$ for odd k and half-integer j). Then, since the collective angular momentum operators do not couple the different spin-J subspaces of the total k-particle space to each other, we have $(\Delta L_\parallel)^2_\phi \geqslant \sum_{J>0} a_J^2 \langle L_y^{(J)}\rangle_{\psi_J} \frac{(\Delta L_\parallel^{(J)})^2_{\psi_J}}{\langle L_y^{(J)}\rangle_{\psi_J}}$. Hence, $(\Delta L_\parallel)^2_\phi \geqslant \sum_{J>0} a_J^2 \langle L_y^{(J)}\rangle_{\psi_J} \zeta_J^2$ follows. Finally, we obtain $\sum_{J>0} a_J^2 \langle L_y^{(J)}\rangle_{\psi_J} \zeta_J^2 \geqslant \zeta_{J_{\max}}^2 \sum_{J>0} a_J^2 \langle L_y^{(J)}\rangle_{\psi_J} = J_{\max} X \zeta_{J_{\max}}^2$, where $J_{\max} = kj$ since $X = \frac{1}{kj}\langle L_y\rangle_\phi = \frac{1}{kj}\sum_{J>0} a_J^2 \langle L_y^{(J)}\rangle_{\psi_J}$ and $\zeta_J^2 \geqslant \zeta_{J'}^2$ for $J \leqslant J'$. The last property can be observed numerically, cf. Fig. 2 (the inset). Due to the concavity of the variance, the statement follows for mixed states. ■

Proof of Observation 3. Let us consider the criterion in Eq. (5) as in Observation 1. Given a certain k, we interpret the degree of violation of the criterion with an estimate of the minimal fraction of particles in $(k+1)$-entangled groups. Let us consider a pure state of N particles $|\Phi_\mathcal{N}\rangle = \bigotimes_{n=1}^{\mathcal{N}} |\phi_n\rangle \otimes |\Psi_{\text{rest}}\rangle$ for some partition that contains $\mathcal{N}$ groups of $k_n \leqslant k$ particles with $\sum_{n=1}^{\mathcal{N}} k_n = M_\mathcal{N}$ and the rest in a collective state $|\Psi_{\text{rest}}\rangle$ of $N - M_\mathcal{N}$ particles that are entangled in groups of $k+1$ or more. For such a state we have $(\Delta J_\parallel)^2_\mathcal{N} \geqslant \sum_{n=1}^{\mathcal{N}} (\Delta L_\parallel)_{\phi_n} \geqslant M_\mathcal{N} j \mathcal{G}_k^{(j)}(\sum_{n=1}^{\mathcal{N}} \langle L_y\rangle_{\phi_n}/M_\mathcal{N} j)$ due to convexity and the fact that $\mathcal{G}_k^{(j)}(X) \geqslant \mathcal{G}_{k_n}^{(j)}(X)$ for $k_n \leqslant k$.

At this point, we assume that $\langle L_y\rangle$ is distributed among the $\mathcal{N}$ groups and the rest of the particles in proportion of the number of particles in these two groups, i.e., $\sum_{n=1}^{\mathcal{N}} \langle L_y\rangle_{\phi_n}/M_\mathcal{N} j = \langle J_y\rangle_\mathcal{N}/Nj$. Hence, we arrive at $(\Delta J_\parallel)^2_\mathcal{N} \geqslant M_\mathcal{N} j \mathcal{G}_k^{(j)}(\langle J_y\rangle_\mathcal{N}/Nj)$. Due to the concavity of the variance and the convexity of $\mathcal{G}_k^{(j)}(X)$ this inequality also holds for mixtures of states of the type $|\Phi_\mathcal{N}\rangle$ with a fixed particle number N, denoted by ϱ_N. Hence, we obtain $(\Delta J_\parallel)^2_{\varrho_N} \geqslant \langle M\rangle_{\varrho_N} j \mathcal{G}_k^{(j)}(\langle J_y\rangle_{\varrho_N}/Nj)$.

Now we consider states $\varrho = \sum_N r_N \varrho_N$, where r_N's are probabilities associated with different numbers of particles N and groupings and define $Q = \langle M\rangle_\varrho/\langle N\rangle_\varrho$, where $\langle N\rangle_\varrho = \sum_N r_N N$ is the average particle number and $\langle M\rangle_\varrho = \sum_N r_N \langle M\rangle_N$. We have $(\Delta J_\parallel)^2_\varrho \geqslant \sum_N r_N (\Delta J_\parallel)^2_N \geqslant \sum_N r_N \langle M\rangle_N j \mathcal{G}_k^{(j)}(\langle J_y\rangle_N/Nj) = Q \sum_N r_N N j \mathcal{G}_k^{(j)}(\langle J_y\rangle_N/Nj)$ and by using the Jensen inequality we arrive at $(\Delta J_\parallel)^2_\varrho \geqslant Q\langle N\rangle_\varrho j \mathcal{G}_k^{(j)}(\langle J_y\rangle_\varrho/\langle N\rangle_\varrho j)$. Using Eqs. (1) and (10), Observation 3 follows. ■

An argument similar to the above can be applied also to the criterion of Sørensen-Mølmer, which states that

$$(\Delta J_z)^2 \geqslant Nj F_J\left(\frac{\langle J_y\rangle}{Nj}\right) \tag{A1}$$

holds in a system of spin-j particles for states with an entanglement depth of at most J/j. Here, $F_J(X)$ is a convex function analogous to $G_J^{\text{sy}}(X)$ [23].

So far, in the derivations we made the assumption that the total polarization splits equally for the different subensembles of atoms. Without such an assumption, first for pure states, we analyze the worst-case scenario in which for a state, such as $|\Phi_\mathcal{N}\rangle$, the polarization splits unequally and state $|\Psi_{\text{rest}}\rangle$ is polarized as much as possible. Hence, we assume $\langle J_y\rangle_{\Psi_{\text{rest}}} = (N - M_\mathcal{N})j$, and it follows that $\sum_{n=1}^{\mathcal{N}} \langle L_y\rangle_{\phi_n} = \langle J_y\rangle_\mathcal{N} - (N - M_\mathcal{N})j$ and consequently $(\Delta J_\parallel)^2_\mathcal{N} \geqslant M_\mathcal{N} j \mathcal{G}_k^{(j)}[(\langle J_y\rangle_\mathcal{N} - (N - M_\mathcal{N})j)/M_\mathcal{N} j]$. Using Eq. (10), we obtain $(\Delta J_\parallel)^2_\mathcal{N} \geqslant \zeta_J^2[\langle J_y\rangle_\mathcal{N} - (N - M_\mathcal{N})j]$. This is clearly valid for mixed states with a varying particle number as $(\Delta J_\parallel)^2_\varrho \geqslant \zeta_J^2[\langle J_y\rangle_\varrho - (\langle N\rangle_\varrho - \langle M\rangle_\varrho)j]$, which can further be rewritten as $Q \leqslant (\xi_\parallel^2/\zeta_J^2 + W - 1)/W$ where $W = \langle N\rangle_\varrho j/\langle J_y\rangle_\varrho$. Then, for a state that is almost fully polarized, i.e., $\langle J_y\rangle_\mathcal{N} \approx \langle N\rangle_\mathcal{N} j$, we recover the statement of Observation 3.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[2] O. Gühne and G. Tóth, Entanglement detection, Phys. Rep. **474**, 1 (2009).

[3] G. Tóth and I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math. Theor. **47**, 424006 (2014).

[4] L. Pezzé, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: Fundamentals and applications in quantum metrology, arXiv:1609.01609.

[5] M. Kitagawa and M. Ueda, Squeezed spin states, Phys. Rev. A **47**, 5138 (1993).

[6] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A **50**, 67 (1994).

[7] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Spin Squeezed Atoms: A Macroscopic Entangled Ensemble Created by Light, Phys. Rev. Lett. **83**, 1319 (1999).

[8] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, Spin Squeezing of Atomic Ensembles via Nuclear-Electronic Spin Entanglement, Phys. Rev. Lett. **101**, 073601 (2008).

[9] C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, and M. A. Kasevich, Squeezed states in a bose-einstein condensate, Science **291**, 2386 (2001).

[10] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Atom-chip-based generation of entanglement for quantum metrology, Nature (London) **464**, 1170 (2010).
[11] J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. Oberthaler, Squeezing and entanglement in a bose–einstein condensate, Nature (London) **455**, 1216 (2008).
[12] M. H. Schleier-Smith, I. D. Leroux, and V. Vuletić, States of an Ensemble of Two-Level Atoms with Reduced Quantum Uncertainty, Phys. Rev. Lett. **104**, 073604 (2010).
[13] I. D. Leroux, M. H. Schleier-Smith, and V. Vuletić, Implementation of Cavity Squeezing of a Collective Atomic Spin, Phys. Rev. Lett. **104**, 073602 (2010).
[14] C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, Nonlinear atom interferometer surpasses classical precision limit, Nature (London) **464**, 1165 (2010).
[15] J. G. Bohnet, K. C. Cox, M. A. Norcia, J. M. Weiner, Z. Chen, and J. K. Thompson, Reduced spin measurement back-action for a phase sensitivity ten times beyond the standard quantum limit, Nat. Photonics **8**, 731 (2014).
[16] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, Magnetic Sensitivity Beyond the Projection Noise Limit by Spin Squeezing, Phys. Rev. Lett. **109**, 253605 (2012).
[17] K. C. Cox, G. P. Greve, J. M. Weiner, and J. K. Thompson, Deterministic Squeezed States with Collective Measurements and Feedback, Phys. Rev. Lett. **116**, 093602 (2016).
[18] O. Hosten, N. J. Engelsen, R. Krishnakumar, and M. A. Kasevich, Measurement noise 100 times lower than the quantum-projection limit using entangled atoms, Nature (London) **529**, 505 (2016).
[19] V. Meyer, M. A. Rowe, D. Kielpinski, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Experimental Demonstration of Entanglement-Enhanced Rotation Angle Estimation using Trapped ions, Phys. Rev. Lett. **86**, 5870 (2001).
[20] R. Auccaise, A. G. Araujo-Ferreira, R. S. Sarthour, I. S. Oliveira, T. J. Bonagamba, and I. Roditi, Spin Squeezing in a Quadrupolar Nuclei Nmr System, Phys. Rev. Lett. **114**, 043604 (2015).
[21] M. W. Mitchell and F. A. Beduini, Extreme spin squeezing for photons, New J. Phys. **16**, 073027 (2014).
[22] A. Sørensen, L.-M. Duan, J. Cirac, and P. Zoller, Many-particle entanglement with bose–einstein condensates, Nature (London) **409**, 63 (2001).
[23] A. S. Sørensen and K. Mølmer, Entanglement and Extreme Spin Squeezing, Phys. Rev. Lett. **86**, 4431 (2001).
[24] B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Detecting Multiparticle Entanglement of Dicke States, Phys. Rev. Lett. **112**, 155304 (2014).
[25] G. Vitagliano, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt, and G. Toth, Entanglement and extreme spin squeezing of unpolarized states, New J. Phys. **19**, 013027 (2017).
[26] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Experimental Entanglement of a Six-Photon Symmetric Dicke State, Phys. Rev. Lett. **103**, 020504 (2009).
[27] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Experimental Realization of Dicke States of up to Six Qubits for Multiparty Quantum Networking, Phys. Rev. Lett. **103**, 020503 (2009).
[28] B. Lücke, M. Scherer, J. Kruse, L. Pezzé, F. Deuretzbacher, P. Hyllus, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, and C. Klempt, Twin matter waves for interferometry beyond the classical limit, Science **334**, 773 (2011).
[29] C. Hamley, C. Gerving, T. Hoang, E. Bookjans, and M. Chapman, Spin-nematic squeezed vacuum in a quantum gas, Nat. Phys. **8**, 305 (2012).
[30] T. M. Hoang, M. Anquez, M. J. Boguslawski, H. M. Bharath, B. A. Robbins, and M. S. Chapman, Adiabatic quenches and characterization of amplitude excitations in a continuous quantum phase transition, Proc. Natl. Acad. Sci. USA **113**, 9475 (2016).
[31] X.-Y. Luo, Y.-Q. Zou, L.-N. Wu, Q. Liu, M.-F. Han, M. K. Tey, and L. You, Deterministic entanglement generation from driving through quantum phase transitions, Science **355**, 620 (2017).
[32] Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, Planar quantum squeezing and atom interferometry, Phys. Rev. A **84**, 022107 (2011).
[33] Q. Y. He, T. G. Vaughan, P. D. Drummond, and M. D. Reid, Entanglement, number fluctuations and optimized interferometric phase measurement, New J. Phys. **14**, 093012 (2012).
[34] G. Puentes, G. Colangelo, R. J. Sewell, and M. W. Mitchell, Planar squeezing by quantum non-demolition measurement in cold atomic ensembles, New J. Phys. **15**, 103031 (2013).
[35] G. Colangelo, F. M. Ciurana, L. C. Bianchet, R. J. Sewell, and M. W. Mitchell, Simultaneous tracking of spin angle and amplitude beyond classical limits, Nature (London) **543**, 525 (2017).
[36] G. Colangelo, F. Martin Ciurana, G. Puentes, M. W. Mitchell, and R. J. Sewell, Entanglement-Enhanced Phase Estimation Without Prior Phase Information, Phys. Rev. Lett. **118**, 233603 (2017).
[37] O. Gühne, G. Tóth, and H. J. Briegel, Multipartite entanglement in spin chains, New J. Phys. **7**, 229 (2005).
[38] O. Gühne, M. Reimpell, and R. F. Werner, Estimating Entanglement Measures in Experiments, Phys. Rev. Lett. **98**, 110502 (2007).
[39] J. Eisert, F. G. S. L. Brandão, and K. M. R. Audenaert, Quantitative entanglement witnesses, New J. Phys. **9**, 46 (2007).
[40] L. Dammeier, R. Schwonnek, and R. F. Werner, Uncertainty relations for angular momentum, New J. Phys. **17**, 093046 (2015).
[41] I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, Optimal witnessing of the quantum fisher information with few measurements, Phys. Rev. A **95**, 032330 (2017).
[42] O. Marty, M. Cramer, G. Vitagliano, G. Toth, and M. B. Plenio, Multiparticle entanglement criteria for nonsymmetric collective variances, arXiv:1708.06986.
[43] Note that the Sørensen-Mølmer condition requires as input just one variance, that must be the one orthogonal to the polarization.
[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.97.020301 for details of the experiment.
[45] G. Colangelo, R. J. Sewell, N. Behbood, F. M. Ciurana, G. Triginer, and M. W. Mitchell, Quantum atom–light interfaces in the gaussian description for spin-1 systems, New J. Phys. **15**, 103007 (2013).