Number-operator–annihilation-operator uncertainty as an alternative for the number-phase uncertainty relation

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(Received 21 January 2010; published 12 May 2010)

We consider a number-operator–annihilation-operator uncertainty as a well-behaved alternative to the number-phase uncertainty relation, and examine its properties. We find a formulation in which the bound on the product of uncertainties depends on the expectation value of the particle number. Thus, while the bound is not a constant, it is a quantity that can easily be controlled in many systems. The uncertainty relation is approximately saturated by number-phase intelligent states. This allows us to define amplitude squeezing, connecting coherent states to Fock states, without a reference to a phase operator. We propose several setups for an experimental verification.

DOI: 10.1103/PhysRevA.81.052108 PACS number(s): 03.65.Fd, 42.50.Dv

I. INTRODUCTION

The finding of a phase operator conjugate to the number operator and construction of number-phase uncertainty relations has an extensive literature [1–3]. However, definition of a Hermitian phase operator for an infinite system, that is, a harmonic oscillator, is not possible, and all the different approaches must make certain compromises.

Historically, the first important contribution was that of Dirac, who introduced the phase observable ϕ based on the decomposition of the annihilation operator as $a = R\exp(i\phi)$ [4]. Assuming that R and ϕ are Hermitian operators, one obtains $R = N^{1/2}$, where $N = a^\dagger a$ is the number operator. Hence, $\exp(i\phi)$ must be equal to

$$E = \sum_{n=0}^{\infty} |n\rangle\,\langle n+1| = (N+1)^{-1/2}a. \tag{1}$$

However, E is not unitary; thus ϕ cannot be Hermitian either.

Several methods have been presented to circumvent the difficulties above. Since there are extensive reviews on the topic [1–3], we cite only the literature that is directly connected to our approach. Susskind and Glogower [5] constructed the Hermitain operators $C = \frac{1}{2}(E + E^\dagger)$ and $S = \frac{1}{2i}(E - E^\dagger)$ to describe the quantum phase. They obtained uncertainty relations with them; however, for the description of the phase two operators were needed. In order to overcome this inconvenience, Lévy-Leblond [6] suggested the use of the non-Hermitian E defined in Eq. (1). He argued that physical quantities could also be represented by non-Hermitian operators, interpreted the meaning of variance for such operators, and wrote down uncertainty relations with N and E. Later, Hermitian phase operators were constructed for finite systems [3,7]. This makes it possible to carry out a calculation for an expression with the phase operator for a finite dimension D, and then take the limit $D \to \infty$, which provides the value corresponding to the infinite-dimensional case. By use of this theoretical background, the number-phase uncertainty relations could be obtained and the states saturating them, called number-phase intelligent states, were identified [8]. The procedure that provides a connection from one number-phase intelligent state to another one with a smaller number variance is called amplitude squeezing [9].

With reference to these ideas, in this paper we choose the two operators to be, not N and $E = (N+1)^{-1/2}a$, but simply N and the annihilation operator a. We present the relation

$$\left((\Delta N)^2 + \frac{1}{4}\right)\left((\Delta a)^2 + \frac{1}{2}\right) \geqslant \frac{\langle N\rangle}{4} + \frac{1}{8}. \tag{2}$$

We will show that states saturating the number-phase uncertainty are very close to saturating Eq. (2). This makes it possible to define amplitude squeezing without reference to a phase operator.

In addition to its connections to quantum optics, this problem is also interesting from the point of view of quantum information theory. A family of uncertainty relations with N and a has already appeared in Ref. [10], and has been used for the detection of quantum entanglement [11–13]. Such uncertainty relations made it possible to construct entanglement conditions with small experimental requirements. Remarkably, these conditions detect non-Gaussian entangled states that cannot be detected based on the first and second moments of the quadrature components [14]. The uncertainty relation Eq. (2) presented in this paper can be seen as a single relation replacing the family of uncertainty relations described in Ref. [10]. For given $\langle N\rangle$, Eq. (2) identifies most of the values for the variances of N and a that are not allowed by quantum physics [15].

The paper is organized as follows. In Sec. II, we discuss how the variance of the annihilation operator can be defined. In Sec. III, we derive the uncertainty relation Eq. (2). In Sec. IV, we discuss the tightness of the uncertainty relation presented. Finally, in Sec. V, we discuss possible physical tests of the proposed uncertainty relation. In the Appendix we present uncertainty relations for two-mode systems.

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II. VARIANCE OF THE ANNIHILATION OPERATOR

In this section, we discuss the definition and the properties of the variance of the annihilation operator. We relate it to quadrature-independent properties of the quantum state.

We define the variance of a non-Hermitian operator A as [6,16]

$$(\Delta A)^2 = \langle A^\dagger A\rangle - \langle A^\dagger\rangle\langle A\rangle. \tag{3}$$

Note that, for non-Hermitian operators, usually we have $(\Delta A)^2 \neq (\Delta A^\dagger)^2$.

Let us now consider the $A = a$ case. $(\Delta a)^2$ is zero *only* for coherent states. The variance $(\Delta a)^2$ measures, in a sense, how close the quantum state is to a coherent state. For this reason, it has been used to study the dynamics of various quantum systems (e.g., see Refs. [17–20]).

Let us now interpret $(\Delta a)^2$ by relating it to the quadrature components.

(i) Let us define the quadrature components as

$$\begin{aligned} x_\beta &= \frac{a e^{+i\beta} + a^\dagger e^{-i\beta}}{\sqrt{2}}, \\ p_\beta &= \frac{a e^{+i\beta} - a^\dagger e^{-i\beta}}{\sqrt{2}i}, \end{aligned} \tag{4}$$

where β is real. Then, one finds that

$$(\Delta a)^2 = \frac{(\Delta x_\beta)^2 + (\Delta p_\beta)^2}{2} - \frac{1}{2}. \tag{5}$$

Hence, $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$ is independent of the choice of the angle β [21].

When the invariance properties of $(\Delta a)^2$ are under discussion, it is instructive to point out its connection to the correlation matrix defined as

$$\Gamma_\beta = \begin{pmatrix} (\Delta x_\beta)^2 & \frac{1}{2}(\langle \Delta x_\beta \Delta p_\beta + \Delta p_\beta \Delta x_\beta\rangle) \\ \frac{1}{2}(\langle \Delta x_\beta \Delta p_\beta + \Delta p_\beta \Delta x_\beta\rangle) & (\Delta p_\beta)^2 \end{pmatrix}. \tag{6}$$

One can obtain $\Gamma_{\beta'}$ from Γ_β through orthogonal transformations. However, the trace of Γ, which equals $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$, remains invariant under such transformations.

Thus, since $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$ is independent of β, it seems to be a good measure of the uncertainty of the orthogonal quadrature components. Note that an alternative measure could be the product $(\Delta x_\beta)^2(\Delta p_\beta)^2$; however, it is not independent of β.

(ii) In another context, $(\Delta a)^2$ can be expressed as

$$(\Delta a)^2 = \frac{1}{2\pi}\int_{\tilde\beta=0}^{2\pi} (\Delta x_{\tilde\beta})^2 d\tilde\beta - \frac{1}{2}. \tag{7}$$

Thus, $(\Delta a)^2$ is connected to the average variance of the quadrature components x_β. That is, if β is chosen randomly between 0 and 2π according to a uniform probability distribution, then $(\Delta a)^2 + \frac{1}{2}$ gives the expectation value of the quadrature variance $(\Delta x_\beta)^2$.

(iii) Finally, let us examine the connection between $(\Delta a)^2$ and important properties of the Wigner function of the quantum state. For the following discussion, as well as in the rest of the paper, we will leave the β subscript, and will use x and p in the sense of x_0 and p_0, respectively. $(\Delta a)^2$ gives information on the sharpness of the peak of the Wigner function $W(x,p)$ of the state, since [22]

$$(\Delta x)^2 + (\Delta p)^2 = \int [(x - \langle x\rangle)^2 + (p - \langle p\rangle)^2] W(x,p)dx\,dp. \tag{8}$$

For states with a non-negative Wigner function (i.e., squeezed coherent states), $2(\Delta a)^2 + 1$ is the sum of the squared widths of the Wigner function in two orthogonal directions. The sharpest peak is obtained for the coherent states for which Eq. (8) is the smallest.

III. UNCERTAINTY RELATION WITH THE NUMBER AND THE ANNIHILATION OPERATORS

In this section, we present a simple derivation of Eq. (2) and relate it to the uncertainty relation with the variances of N and E. We also discuss how to improve the relation Eq. (2).

We start from the two Heisenberg uncertainty relations

$$\begin{aligned} (\Delta N)^2(\Delta p)^2 &\geqslant \tfrac{1}{4}|\langle x\rangle|^2, \\ (\Delta N)^2(\Delta x)^2 &\geqslant \tfrac{1}{4}|\langle p\rangle|^2, \end{aligned} \tag{9}$$

where we used the fact that for operators A and B we have $(\Delta A)^2(\Delta B)^2 \geqslant \frac{1}{4}|\langle[A,B]\rangle|^2$ [22]. Summing the two inequalities of Eq. (9) and using Eq. (5) and $|\langle p\rangle|^2 + |\langle x\rangle|^2 = 2|\langle a\rangle|^2$, one obtains the following uncertainty relation with N and a:

$$(\Delta N)^2\big((\Delta a)^2 + \tfrac{1}{2}\big) \geqslant \tfrac{1}{4}|\langle a\rangle|^2. \tag{10}$$

From Eq. (10) it follows that, knowing $\langle a\rangle$, which determines the "center" of the Wigner function $W(x,p)$ of the state, and $(\Delta a)^2$, which is based on the the width of the Wigner function in two orthogonal directions, we can obtain a lower bound for the particle number fluctuation.

The bound in the uncertainty relation Eq. (10) is not a constant: it depends on $\langle a\rangle$, which is zero for a wide class of states. It would be meaningful to find a similar relation with a constant bound, or at least with a bound depending on a quantity that is easily measurable and controllable.

We now construct a relation in which the bound depends on $\langle N\rangle$ rather than on $\langle a\rangle$. For that, we add $[(\Delta a)^2 + \frac{1}{2}]/4$ to both sides of Eq. (10), and, using $(\Delta a)^2 = \langle N\rangle - |\langle a\rangle|^2$, we obtain Eq. (2). The right-hand side of Eq. (2) is minimal for the vacuum $|0\rangle$. In all other cases, the right-hand side is greater than $\frac{1}{8}$; thus the uncertainty finds some part of the $(\Delta a)^2$-$(\Delta N)^2$ plane inaccessible for quantum states.

Next, we relate Eq. (2) to the uncertainty relation with N and E. For that, we determine the form of Eq. (2) for the case of large N and $(\Delta N)^2 \ll \langle N\rangle^2$. In this case, $a \approx \sqrt{\langle N\rangle}E$ and we obtain

$$(\Delta N)^2(\Delta E)^2 \gtrsim \tfrac{1}{4}[1 - (\Delta E)^2], \tag{11}$$

which is in accordance with the results of Ref. [6],

$$(\Delta N)^2(\Delta E)^2 \geqslant \tfrac{1}{4}[1 - (\Delta E)^2 - \langle P^{(0)}\rangle], \tag{12}$$

where $P^{(0)} = |0\rangle\langle 0|$.

Finally, Eq. (2) can be improved by use of the Robertson-Schrödinger inequalities [22]. First, they can be used to improve the two uncertainty relations in Eq. (9). Then, after steps similar to the previous ones, we obtain

$$\left((\Delta N)^2+\frac{1}{4}\right)\left((\Delta a)^2+\frac{1}{2}\right)\geqslant\frac{\langle N\rangle}{4}+\frac{1}{8}+\frac{1}{4}|\langle\{\Delta N,\Delta a\}_+\rangle|^2, \tag{13}$$

where $\{A,B\}_+ = AB + BA$ is the anticommutator.

IV. TIGHTNESS OF THE INEQUALITY

In this section, we investigate the tightness of Eq. (2) and look for quantum states that are close to saturating it. We also discuss the result that the states saturating the left-hand side of Eq. (2) interpolate between coherent states and Fock states.

Our inequality does not contain the highest possible lower bound. The reason for that is that we constructed Eq. (2) by summing the two uncertainty relations in Eq. (9). While the relation Eq. (10) is valid, it is not tight, since the two uncertainty relations in Eq. (9) are saturated by different states. Thus, the tightness of the bound in Eq. (2) must be verified.

In Fig. 1, we plotted the points corresponding to values of $[(\Delta a)^2,(\Delta N)^2]$ that saturate Eq. (2) for $\langle N\rangle = 25$. All points below this line violate the relation Eq. (2). For Fock states,

$$(\Delta a)^2_{\text{Fock}} = \langle N\rangle, \quad (\Delta N)^2_{\text{Fock}} = 0. \tag{14}$$

Hence, Fock states saturate Eq. (2). For coherent states we have

$$(\Delta a)^2_{\text{coh}} = 0, \quad (\Delta N)^2_{\text{coh}} = \langle N\rangle. \tag{15}$$

For $(\Delta a)^2 = 0$, the particle number variance saturating Eq. (2) is $(\Delta N)^2 = \frac{1}{2}\langle N\rangle$. This already shows that the lower bound in Eq. (2) cannot be optimal, because for $(\Delta a)^2 = 0$ there is no quantum state with a smaller particle number variance than $\langle N\rangle$. States minimizing $(\Delta N)^2$ for $0 < (\Delta a)^2 < \langle N\rangle$, in a sense, interpolate between coherent states and Fock states.

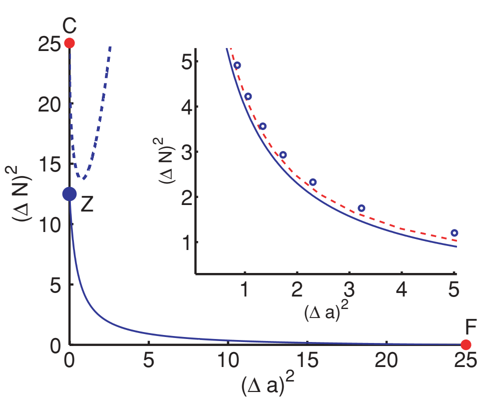

FIG. 1. (Color online) Numerical test of the inequality (2). F refers to the Fock state $|n = 25\rangle$, C to the coherent state $|\alpha = 5\rangle$, and Z to the point that saturates inequality (2) for $(\Delta a)^2 = 0$. Solid line: Boundary of the region defined by Eq. (2) for $\langle N\rangle = 25$. All points below this line correspond to aphysical $(\Delta a)^2$-$(\Delta N)^2$ values. Dashed line: Points corresponding to squeezed coherent states. The equation of the curve is given in Eq. (23). Inset: Solid line: Boundary of the region defined by Eq. (2). Dashed line: Points corresponding to states with a Gaussian wave vector Eq. (16). Circles: States corresponding to photon-added coherent states, Eq. (27).

We will now examine the tightness of Eq. (2) numerically through choice of appropriate trial states. Our search can be simplified by noting that it is sufficient to search over wave vectors with non-negative real elements. To see this, let us consider a state of the form $|\Psi\rangle = \sum_n |c_n|e^{i\phi_n}|n\rangle$. One finds that, if all angles ϕ_n are set to zero, $(\Delta N)^2$ and $\langle N\rangle$ do not change. On the other hand, $|\langle a\rangle|$ cannot decrease. Hence, $(\Delta a)^2 = \langle N\rangle - |\langle a\rangle|^2$ cannot increase. Thus, it is sufficient to search over states with all $\phi_n = 0$. Moreover, a state with $\phi_n = \text{const}\times n$ will give the same values for $(\Delta N)^2,\langle N\rangle$, and $(\Delta a)^2$ as does a state with $\phi_n = 0$.

A. Gaussian wave vector

Let us consider states with a Gaussian state vector

$$|N_0,\Delta\rangle = \frac{1}{C}\sum_n \exp\left(-\frac{(n-N_0)^2}{4\Delta^2}\right)|n\rangle\,, \tag{16}$$

where C is for normalization. For such states, $\langle N\rangle \approx N_0$ and $(\Delta N)^2 \approx \Delta^2$. In the inset of Fig. 1, the dashed line corresponds to states of the form Eq. (16), while the solid line corresponds to points saturating Eq. (2). It can be seen that states of the form Eq. (16) are close to saturating Eq. (2). In Fig. 2, we plotted the relative difference between the left- and the right-hand sides of Eq. (2) for particular values of Δ and N_0.

Finally, Fig. 3 shows the distance in the $(\Delta a)^2$-$(\Delta N)^2$ plane of the points corresponding to states of the form Eq. (16) from the curve corresponding to states that saturate Eq. (2) [23]. The distance does not grow with N and remains smaller than 0.15. This means that for large N, for which $(\Delta a)^2$ and $(\Delta N)^2$ cannot be measured with an accuracy of 0.15, the states Eq. (16) are indistinguishable from the intelligent states of the uncertainty relation Eq. (2).

The states Eq. (16) are the subset of the number-phase intelligent states called $|g'\rangle$ presented in Ref. [8]. There, the coefficients of $|n\rangle$ have a Gaussian dependence on n, just as in Eq. (16); however, the phase of the coefficients is not

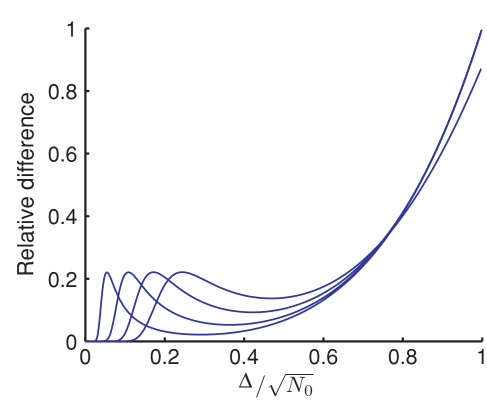

FIG. 2. (Color online) Numerical test of the inequality Eq. (2). The difference between the left- and right-hand sides of Eq. (2) divided by the right-hand side is shown for states Eq. (16) with a Gaussian wave vector for (from left to right) $N_0 = 100$, 25, 10, and 5.

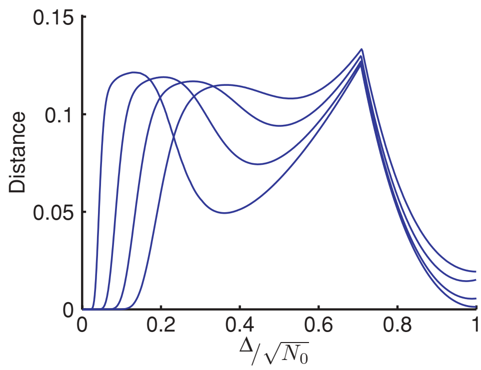

FIG. 3. (Color online) Distance of the points corresponding to states with a Gaussian wave vector Eq. (16) from the curve corresponding to states that saturate Eq. (2) for (from left to right) $N_0 = 100$, 25, 10, and 5. Δ and N_0 are the parameters of the state Eq. (16).

zero but has a linear dependence on n. As we have already discussed, a state vector with a phase with a linear dependence on n has the same values for $(\Delta a)^2$,$(\Delta N)^2$, and $\langle N\rangle$ as a state with zero phase. Thus, all the $|g'\rangle$ states presented in Ref. [8] are very close to saturating Eq. (2). Hence, our inequality makes it possible to define number-phase intelligent states and amplitude squeezing [9] without reference to a phase operator.

There are other states known to be number-phase intelligent states [24,25]. We now consider the states presented in Ref. [26]. They are defined as the superposition of coherent states on a circle,

$$|\alpha_0,u\rangle \propto \int_{-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}u^2\phi^2 - i\delta\phi\right)|\alpha_0 e^{i\phi}\rangle d\phi, \quad (17)$$

where $\delta = \alpha_0^2$. The overlap with Fock states is

$$\begin{aligned}\langle\alpha_0,u|n\rangle &\propto \frac{\alpha_0^n}{\sqrt{n!}}\exp\left(-\frac{(n-\delta)^2}{2u^2}\right)\\ &= \langle\alpha_0|n\rangle \exp\left(-\frac{(n-\delta)^2}{2u^2}\right).\end{aligned} \quad (18)$$

The second expression stresses the fact that we have the overlap of a coherent state $|\alpha_0\rangle$ and a Fock state $|n\rangle$, multiplied by a Gaussian centered around α_0^2, that is, the expectation value of the particle number for $|\alpha_0\rangle$. Thus, $|\alpha_0,u\rangle$ has an almost Gaussian wave vector for large N in the number basis. Hence, these states give similar results numerically for our uncertainty relation Eq. (2) as does Eq. (16).

B. Squeezed coherent states

It is natural to ask to what extent squeezed coherent states approach the curve defined by Eq. (2). Squeezed coherent states can be obtained from the vacuum state as [27,28]

$$|\alpha,\zeta\rangle = D(\alpha)S(\zeta)|0\rangle\,, \quad (19)$$

where D is the displacement operator and S is the squeezing operator. Next, we use the relationships

$$\begin{aligned}D^\dagger(\alpha)aD(\alpha) &= a+\alpha,\\ S^\dagger(\zeta)aS(\zeta) &= a\cosh(s) - a^\dagger e^{i\vartheta}\sinh(s),\end{aligned} \quad (20)$$

where $\zeta = se^{i\vartheta}$. Hence, with $\alpha = |\alpha|e^{i\theta}$, we obtain

$$\begin{aligned}\langle N\rangle_{|\alpha,\zeta\rangle} &= \sinh^2(s) + |\alpha|^2,\\ (\Delta a)^2_{|\alpha,\zeta\rangle} &= \sinh^2 s,\\ (\Delta N)^2_{|\alpha,\zeta\rangle} &= |\alpha|^2\,[\cosh(2s) - \sinh(2s)\cos(2\theta-\vartheta)]\\ &\quad + 2\sinh^2(s)[1+\sinh^2(s)].\end{aligned} \quad (21)$$

For given $|\alpha|$ and s, the variance $(\Delta N)^2$ in Eq. (21) is minimal if $\cos(2\theta-\vartheta) = 1$. This is fulfilled, for example, if $\theta = \vartheta = 0$, that is, both ζ and α are real and non-negative. Hence,

$$\begin{aligned}(\Delta N)^2_{\min}(|\alpha|,s) &= |\alpha|^2[\cosh(2s) - \sinh(2s)]\\ &\quad + 2\sinh^2(s)[1+\sinh^2(s)].\end{aligned} \quad (22)$$

Based on Eq. (22), we obtain the smallest possible $(\Delta N)^2$ for squeezed coherent states, for given $(\Delta a)^2$ and $\langle N\rangle$, as

$$\begin{aligned}(\Delta N)^2_{\min} &= [\langle N\rangle - (\Delta a)^2][\sqrt{1+(\Delta a)^2} - \sqrt{(\Delta a)^2}]^2\\ &\quad + 2(\Delta a)^2[1+(\Delta a)^2].\end{aligned} \quad (23)$$

The dashed curve in Fig. 1 corresponds to Eq. (23).

Let us interpret this result. Since $(\Delta a)^2$ and $(\Delta N)^2$ are invariant under a rotation around the origin in the x-p plane, we can start from coherent states $|\alpha\rangle$ with a real and positive α. Then the state we considered for the curve Eq. (23) corresponds to squeezing of the x quadrature component, which is called "number squeezing" in the literature (e.g., see Ref. [28]), and it reduces the number variance for a small amount of squeezing. Thus, for small squeezing Eq. (23) is not far from the bound given by Eq. (2). With further squeezing, the number variance starts to grow. Thus, for large $(\Delta a)^2$, there are no squeezed coherent states giving an almost minimal particle number variance, and one has to look for non-Gaussian states for that. As a by-product of our discussion, note that the non-Gaussianness of quantum states can be verified by measuring only $(\Delta a)^2$ and $(\Delta N)^2$.

C. Displaced Fock states

Displaced Fock states are defined as [29]

$$|\alpha,n\rangle = D(\alpha)\,|n\rangle\,. \quad (24)$$

Using Eq. (20), we obtain

$$\begin{aligned}\langle N\rangle_{|\alpha,n\rangle} &= n + |\alpha|^2,\\ (\Delta N)^2_{|\alpha,n\rangle} &= (2n+1)|\alpha|^2,\\ (\Delta a)^2_{|\alpha,n\rangle} &= n.\end{aligned} \quad (25)$$

Hence, for displaced Fock states we get the equation

$$(\Delta N)^2 = [2(\Delta a)^2 + 1][\langle N\rangle - (\Delta a)^2], \quad (26)$$

where $(\Delta a)^2$ must be a non-negative integer. It is fulfilled by both Fock states and coherent states. Other points in the $(\Delta a)^2$-$(\Delta N)^2$ plane satisfying Eq. (26) are very far from saturating Eq. (2).

D. Photon-added coherent states

Photon-added coherent states are defined as [30]

$$|\alpha,m\rangle \propto (a^\dagger)^m\,|\alpha\rangle\,. \quad (27)$$

They are close to saturating Eq. (2), as can be seen in Fig. 1.

E. Eigenstates of $a^\dagger a$ + const × a

According to Heisenberg's method, states that minimize the uncertainty product $(\Delta X)^2(\Delta Y)^2$ for Hermitian X and Y with a constant commutator are the eigenstates of $X + icY$, where c is some constant [31]. While in Eq. (2) we do not have the product of the uncertainties of two Hermitian observables, this method can still give us the idea of considering the states $|d,k\rangle$ defined through the eigenvalue equation

$$(a^\dagger a + da)\,|d,k\rangle = k\,|d,k\rangle, \tag{28}$$

where d and k are constants. Writing $|d,k\rangle$ as $\sum_k c'_n\,|n\rangle$, we obtain

$$c'_{n+1} = \frac{(k-n)}{d\sqrt{n+1}}c'_n \tag{29}$$

for the coefficients. Equation (29) leads to a normalizable wave vector only if k is a non-negative integer. In this case, $c_l = 0$ for all $l > k$. Numerical evidence suggests that states $|d,k\rangle$ are close to saturating Eq. (2), but they are inferior to the states given by Eq. (16).

F. States minimizing $(\Delta N)^2$ for given $(\Delta a)^2$ and $\langle N\rangle$

Let us now look for states that minimize $(\Delta N)^2$ for given $(\Delta a)^2$ and $\langle N\rangle$. For that, we follow an approach similar to the one presented in Ref. [31]. Since $(\Delta a)^2 = \langle N\rangle - \frac{1}{2}(\langle x\rangle^2 + \langle p\rangle^2)$, this task can be reformulated as a search for the states that minimize $(\Delta N)^2$ for given $\langle x\rangle$, $\langle p\rangle$, and $\langle N\rangle$. Let us write the state as $|\Phi\rangle = \sum_k c''_n\,|n\rangle$. Hence, we have to look for the minimum of the function

$$\begin{aligned} &f(\langle\Phi|,|\Phi\rangle,\lambda_N,\lambda_p,\lambda_x)\\ &\quad= \langle N^2\rangle - N_0^2 + \lambda_x(\langle x\rangle - x_0) + \lambda_p(\langle p\rangle - p_0)\\ &\qquad+ \lambda_N(\langle N\rangle - N_0), \end{aligned} \tag{30}$$

where λ_k are Lagrange multipliers. Note that we do not include explicitly the $\langle\Phi|\Phi\rangle = 1$ condition in the function f. The minimum is given by one of the critical points for which all derivatives are zero. Equation (30) can be rewritten as

$$f(\langle\Phi|,|\Phi\rangle,\lambda_N,\lambda_p,\lambda_x) = \langle O(\lambda_N,\lambda_p,\lambda_x)\rangle_{|\Phi\rangle}, \tag{31}$$

where O is defined as

$$\begin{aligned} &O(\lambda_N,\lambda_p,\lambda_x)\\ &\quad= \lambda_N a^\dagger a + (a^\dagger a)^2 + \left(\frac{\lambda_x + i\lambda_p}{\sqrt{2}}\right)a + \left(\frac{\lambda_x - i\lambda_p}{\sqrt{2}}\right)a^\dagger. \end{aligned} \tag{32}$$

We have to look for $\{|\Phi^{(k)}\rangle,\lambda_N^{(k)},\lambda_x^{(k)},\lambda_p^{(k)}\}$ that minimize Eq. (31). It is easy to see that $|\Phi^{(k)}\rangle$ must minimize $\langle O(\lambda_N^{(k)},\lambda_p^{(k)},\lambda_x^{(k)})\rangle$. Hence, states $|\Phi^{(k)}\rangle$ must be the eigenstates of the operator $O(\lambda_N^{(k)},\lambda_p^{(k)},\lambda_x^{(k)})$ with the smallest eigenvalue (i.e., they have to be "ground states"). Note that the operator given in Eq. (32) appears as a system Hamiltonian in self-consistent calculations for the Bose-Hubbard model based on the Gutzwiller ansatz [32,33].

V. DISCUSSION

Let us discuss the use of quantum states that minimize $(\Delta N)^2$ for given $\langle N\rangle$ and $(\Delta a)^2$. They present a trade-off between two requirements: the smallest possible variance of a randomly chosen quadrature component and the smallest possible particle number variance. In a sense, they are similar to states minimizing $(\Delta N)^2$ for given $(\Delta\phi)^2$. The latter present a trade-off between the smallest possible variances for phase measurements and for particle number measurements.

Clearly, the right-hand side of Eq. (2) is not a constant, but it is a quantity that can be controlled easily in many systems. Moreover, note that the measurement of $(\Delta a)^2$ does not require measurement of variances of x and p if we use $(\Delta a)^2 = \langle N\rangle - |\langle a\rangle|^2 = \langle N\rangle - \frac{1}{2}(\langle x\rangle^2 + \langle p\rangle^2)$.

A single trapped ion seems to be a good candidate for testing our inequalities and realizing quantum states that saturate them [34–36]. For a trapped ion, x and p are the physical position and momentum coordinates, and N determines the energy of the ion.

The uncertainty relation Eq. (2) can also be verified experimentally in a single-mode electromagnetic field. The two orthogonal quadrature components can be measured, for example, with homodyne detection [37]. The result is not influenced by which two orthogonal components we choose to measure.

Bose-Einstein condensates of alkali-metal atoms seem to be also a possible candidate for experiments [35,38]. It is usual to talk about number squeezing in multiwell Bose-Einstein condensates in the sense that an increase in the barrier height between the wells decreases the number fluctuation within the wells [39]. Here, one has to note that for cold atoms the particle number is conserved. Because of that, for a single bosonic mode of cold atoms, superpositions of states with different particle numbers are not allowed. For this reason, for pure states $(\Delta N)^2 = 0$. It is possible to mix states with different particle numbers, making $(\Delta N)^2 > 0$. However, even for such states $\langle a\rangle = 0$ and $(\Delta a)^2 = \langle N\rangle$, which makes our inequalities trivial in such systems.

While we cannot create particles in a single mode, we can move particles from one mode to another one. Thus, it is instructive to consider two-mode systems of cold atoms. The two modes can be realized with atoms in a double well or with a single Bose-Einstein condensate of two-state atoms. Let us denote the annihilation operators of the two modes by a_1 and a_2, respectively. The corresponding particle numbers are N_1 and N_2. If $\langle N_1\rangle \ll \langle N_2\rangle$ and $(\Delta N_2)^2 \ll \langle N_2\rangle^2$, then with the substitution

$$\begin{aligned} a &\to \frac{a_1 a_2^\dagger}{\sqrt{\langle N_2\rangle}},\\ N &\to N_1 \end{aligned} \tag{33}$$

the uncertainty relations Eqs. (10) and (2) can be tested. In the Appendix we present relations that do not require such approximations.

Finally, in the statistical physics of bosonic systems, $\langle\Psi(x)\rangle$, that is, the expectation value of the field operator, plays the role of the order parameter. In this context, our findings present a quantitative relationship between the variance of the field operator and the variance of the particle density $\Psi(x)^\dagger\Psi(x)$.

VI. SUMMARY

We constructed uncertainty relations with the particle number and the annihilation operator. The variance of the latter describes the uncertainty in the phase space, and is independent of the absolute phase of the quadrature components. We proposed quantum optical systems in which our inequality could be tested.

ACKNOWLEDGMENTS

We thank I. L. Egusquiza, J. J. García-Ripoll, and M. W. Mitchell for fruitful discussions. We thank P. Adam for interesting discussions on amplitude squeezing. We thank the National Research Fund of Hungary OTKA (Contract No. T049234), and the Spanish MEC (Ramon y Cajal Program, Consolider-Ingenio 2010 project "QOIT," Project No. FIS2009-12773-C02-02) for support. I.U.L. acknowledges the support of a Ph.D. grant of the Basque Government and thanks the Department of Theoretical Physics, where most of the research for this paper was completed in a summer project, for hospitality.

APPENDIX: UNCERTAINTY RELATIONS FOR TWO-MODE SYSTEMS

For the two-mode system, inequalities similar to Eqs. (10) and (2) can be found using the Schwinger representations of the angular momentum operators

$$J_l = \frac{1}{2}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}^T \sigma_l \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{A1}$$

for $l = x,y,z$, where σ_l are the Pauli spin matrices. Let us define an operator that is an analog of a in the two-mode system as

$$\tilde{a} = J_x - iJ_y \equiv a_1 a_2^\dagger. \tag{A2}$$

With this definition, we have

$$\begin{aligned} (|\Delta\tilde{a}|^2) &= \tfrac{1}{2}[(\Delta\tilde{a})^2 + (\Delta\tilde{a}^\dagger)^2] = (\Delta J_x)^2 + (\Delta J_y)^2, \\ |\langle\tilde{a}\rangle|^2 &= \langle J_x\rangle^2 + \langle J_y\rangle^2. \end{aligned} \tag{A3}$$

Using Eq. (A3) and the Heisenberg uncertainty relation $(\Delta J_k)^2(\Delta J_l)^2 \geqslant \frac{1}{4}|\langle J_m\rangle|^2$, we obtain the analog of Eq. (10),

$$(\Delta N_1)^2(|\Delta\tilde{a}|^2) \geqslant \tfrac{1}{4}|\langle\tilde{a}\rangle|^2. \tag{A4}$$

Adding $\frac{1}{4}(|\Delta\tilde{a}|^2)$ to both sides of Eq. (A4) and using

$$J_x^2 + J_y^2 = \tfrac{1}{2}(N_1+1)(N_2+1) - \tfrac{1}{2}, \tag{A5}$$

we obtain an analog of Eq. (2),

$$\big((\Delta N_1)^2 + \tfrac{1}{4}\big)(|\Delta\tilde{a}|^2) \geqslant \tfrac{1}{8}\langle(N_1+1)(N_2+1)\rangle - \tfrac{1}{8}. \tag{A6}$$

As we mentioned previously, number squeezing with Bose-Einstein condensates in a double well can occur if the barrier between the wells increases [39]. Equation (2) bounds the number variance of a well in such systems [40].

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