(http://iopscience.iop.org/1367-2630/19/1/013027)

New Journal of Physics

The open access journal at the forefront of physics

CrossMark

PAPER

Entanglement and extreme spin squeezing of unpolarized states

Giuseppe Vitagliano[1], **Iagoba Apellaniz**[1], **Matthias Kleinmann**[1], **Bernd Lücke**[2], **Carsten Klempt**[2] and **Géza Tóth**[1,3,4]

1   Department of Theoretical Physics, University of the Basque Country UPV/EHU, PO Box 644, E-48080 Bilbao, Spain
2   Institut für Quantenoptik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
3   IKERBASQUE, Basque Foundation for Science, E-48013 Bilbao, Spain
4   Wigner Research Centre for Physics, Hungarian Academy of Sciences, PO Box 49, H-1525 Budapest, Hungary

E-mail: gius.vitagliano@gmail.com and toth@alumni.nd.edu

Abstract

We present criteria to detect the depth of entanglement in macroscopic ensembles of spin-j particles using the variance and second moments of the collective spin components. The class of states detected goes beyond traditional spin-squeezed states by including Dicke states and other unpolarized states. The criteria derived are easy to evaluate numerically even for systems of very many particles and outperform past approaches, especially in practical situations where noise is present. We also derive analytic lower bounds based on the linearization of our criteria, which make it possible to define spin-squeezing parameters for Dicke states. In addition, we obtain spin squeezing parameters also from the condition derived in (Sørensen and Mølmer 2001 *Phys. Rev. Lett.* **86** 4431). We also extend our results to systems with fluctuating number of particles.

1. Introduction

With an interest towards fundamental questions in quantum physics, as well as applications, larger and larger entangled quantum systems have been realized with photons, trapped ions, and cold atoms [1–11]. Entanglement is needed for certain quantum information processing tasks [12, 13], and it is also necessary to reach the maximum sensitivity in a wide range of interferometric schemes in quantum metrology [14]. Hence, the verification of the presence of entanglement is a crucial but exceedingly challenging task, especially in an ensemble of many, say 10^3 or 10^{12} particles [5–11]. Moreover, in such experiments it is not sufficient to claim that 'the state is entangled', we need also to know how entangled the system is. Hence, quantifying entanglement in large ensembles has recently been at the center of attention. In several experiments the *entanglement depth* (i.e., the minimal number of mutually entangled particles consistent with the measurement data) was determined, reaching to the thousands [7–11].

In the many-particle case, especially in large ensembles of cold atoms, it is typically very difficult or even impossible to address the particles individually, while measuring collective quantities is still feasible. In this context, one of the most successful approaches to detect entanglement is based on the criterion [15]

$$\xi_s^2 := N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geqslant 1, \tag{1}$$

where N is the number of the spin-1/2 particles, $J_l = \sum_{n=1}^N j_l^{(n)}$ for $l = x, y, z$ are the collective spin components, and $j_l^{(n)}$ are single particle spin components acting on the nth particle. Every multiqubit state that violates (1) must be entangled [15]. The criterion (1) is best suited for states with a large collective spin in the $(\hat{y}, \hat{z})$-plane and a small variance $(\Delta J_x)^2$ in the orthogonal direction. For such states the variance of a spin component is reduced below what can be achieved with fully polarized spin-coherent states, hence they have been called *spin squeezed* in the context of metrology [16, 17].

As a generalization of (1), a criterion has also been derived by Sørensen and Mølmer [18] to detect the depth of entanglement of spin-squeezed states in an ensemble of particles with a spin j. For the criterion, we have to

consider a subgroup of $k \leqslant N$ particles and define its total spin as

$$J = kj. \tag{2}$$

We also need to define a function F_J via a minimization over quantum states of such a group as

$$F_J(X) := \frac{1}{J} \min_{\varrho : \frac{1}{J}\langle L_z \rangle_\varrho = X} (\Delta L_x)^2_\varrho, \tag{3}$$

where L_l are the spin components of the group. In practice, the minimum will be the same if we carry out the minimization over states of a single particle with a spin J [19]. Then, for all pure states with an entanglement depth of at most k

$$(\Delta J_x)^2 \geqslant Nj F_J\left(\frac{\langle J_z \rangle}{Nj}\right) \tag{4}$$

holds. It is easy to see that (4) is valid even for mixed states with an entanglement depth of at most k since the variance is concave in the state and $F_J(X)$ is convex[5]. Thus, every state that violates (4) must have a depth of entanglement of $(k + 1)$ or larger. It is important to stress that the criterion (4) provides a tight lower bound on $(\Delta J_x)^2$ based on $\langle J_z \rangle$. Spin squeezing has been demonstrated in many experiments, from cold atoms [7, 20–26] to trapped ions [27], magnetic systems [28] and photons [29], and in many of these experiments even multipartite entanglement has been detected using the condition (4) [7, 23–26, 29].

Recently, the concept of spin squeezing has been extended to unpolarized states [30–34]. In particular, Dicke states are attracting increasing attention, since their multipartite entanglement is robust against particle loss, and they can be used for high precision quantum metrology [8]. Dicke states are produced in experiments with photons [35, 36] and Bose–Einstein condensates [8, 37, 38]. Suitable criteria to detect the depth of entanglement of Dicke states have also been derived. However, either they are limited to spin-$1/2$ particles [8, 39] or they do not give a tight lower bound on $(\Delta J_x)^2$ based on the expectation value measured for the criterion, concretely, $\langle J_y^2 + J_z^2 \rangle$ [40].

In this paper, we present a general condition that: (i) provides a lower bound on the entanglement depth, (ii) is applicable to spin-j systems, for any j, (iii) works both for spin-squeezed states and Dicke states, and, (iv) is close to provide a tight bound in the sense mentioned above in the large particle number limit. Such a criterion can be applied immediately, for instance, in experiments producing Dicke states in spinor condensates [41].

We now summarize the main results of our paper. With a method similar to the one used for obtaining equation (4), we show that the condition

$$(\Delta J_x)^2 \geqslant Nj G_J\left(\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1)}{N(N - k)j^2}\right) \tag{5}$$

holds for states with an entanglement depth of at most k of an ensemble of N spin-j particles, where we introduced the notation

$$G_J : X \mapsto F_J(\sqrt{X}), \tag{6}$$

with $F_J(X)$ defined as in equation (3) and $J = kj$ as in (2). Our approach is motivated by the fact that equation (4) fails to be a good criterion for mixed states with a low polarization $\langle J_y \rangle^2 + \langle J_z \rangle^2 \ll N^2 j^2$. Thus, we consider the second moments $\langle J_y^2 + J_z^2 \rangle$ instead, which are still large for many useful unpolarized quantum states, such as Dicke states. Using the second moments is advantageous even for states with a large spin polarization since criteria with second moments are more robust to noise, which will be demonstrated later on concrete examples. We also analyze the performance of our condition compared to other criteria in the literature.

In general, the function $G_J(X)$ appearing on the right-hand side of (5) has to be evaluated numerically. However, due to its convexity properties we can bound it from below with the two lowest non-trivial orders of its Taylor expansion around $X = 0$, yielding a spin-squeezing parameter similar to the one defined in (1). While states saturating (5) determine a curve in the $(\langle J_y^2 + J_z^2 \rangle, (\Delta J_x)^2)$-plane, such an analytic condition corresponds to tangents to this curve. Hence, we will refer to it as a linear criterion. Such a criterion for states with an entanglement depth k or smaller is given by the inequality

$$\xi^2 := (kj + 1)\frac{2(N - k)j(\Delta J_x)^2}{\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1)} \geqslant 1, \tag{7}$$

where we require that kj is an integer. A similar condition can be obtained from the Sørensen–Mølmer criterion (4) as

---

[5] The convexity of $F_J(X)$ is observed numerically [18]. In case the right-hand side of (3) results in a non-convex function in X, then the convex hull of the right-hand side of (3) must be used in the place of $F_J(X)$.

$$\xi_{\mathrm{SM}}^2 := (kj+1)\frac{2Nj(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geqslant 1, \tag{8}$$

again requiring that kj is integer. A direct comparison between ξ^2 and ξ_{SM}^2 shows that the former is more suitable for detecting the depth of entanglement of unpolarized states, such as Dicke states. Note also the similarity between (8) and the original criterion for spin-1/2 particles (1). All these criteria are also generalized to the case when the particle number is not fixed, following [19].

Our paper is organized as follows. In section 2, we discuss how to evaluate our criteria numerically, while we also discuss cases where analytical formulas can be used instead of numerics. In section 3, we derive our nonlinear entanglement criterion. In section 4, we present linear criteria leading to new spin-squeezing parameters. In section 5, we compare our entanglement conditions to other conditions existing in the literature. Finally, in section 6, we discuss how to generalize our methods to the case of a fluctuating number of particles.

2. Numerical computation of $G_J(X)$

Before describing how to obtain $F_J(X)$ and $G_J(X)$ numerically, we define some notions necessary for our discussion. We distinguish various levels of multipartite entanglement based on the following definitions. A pure quantum state is k-producible if it can be written as

$$|\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle \otimes \cdots \otimes |\psi^{(M)}\rangle, \tag{9}$$

where $|\psi^{(l)}\rangle$ are states of $k_l \leqslant k$ particles, and M stands for the number of particle groups. A mixed quantum state is k-producible, if it can be written as a mixture of pure k-producible states. Clearly, 1-producible states are separable states. A state that is not k-producible is called $(k+1)$-entangled. The entanglement depth is $k+1$ whenever the state is $(k+1)$-producible but not k-producible [18, 42].

Next, we will show a simple method to calculate $F_J(X)$ and $G_J(X)$. We will discuss both numerical and analytical approaches. Knowing the properties of these functions is necessary to prove later the relation (5). For an integer J, the function $F_J(X)$ given in (3) can be efficiently computed for some interval of X as follows [18]. We just need to calculate the ground states $|\phi_\lambda\rangle$ of the Hamiltonian
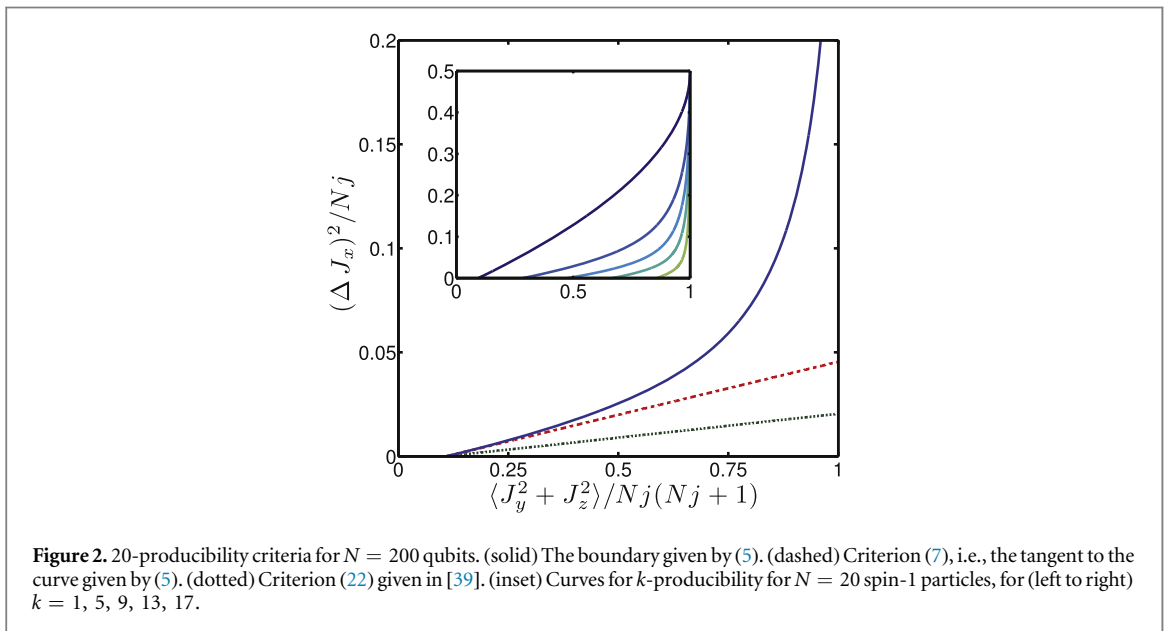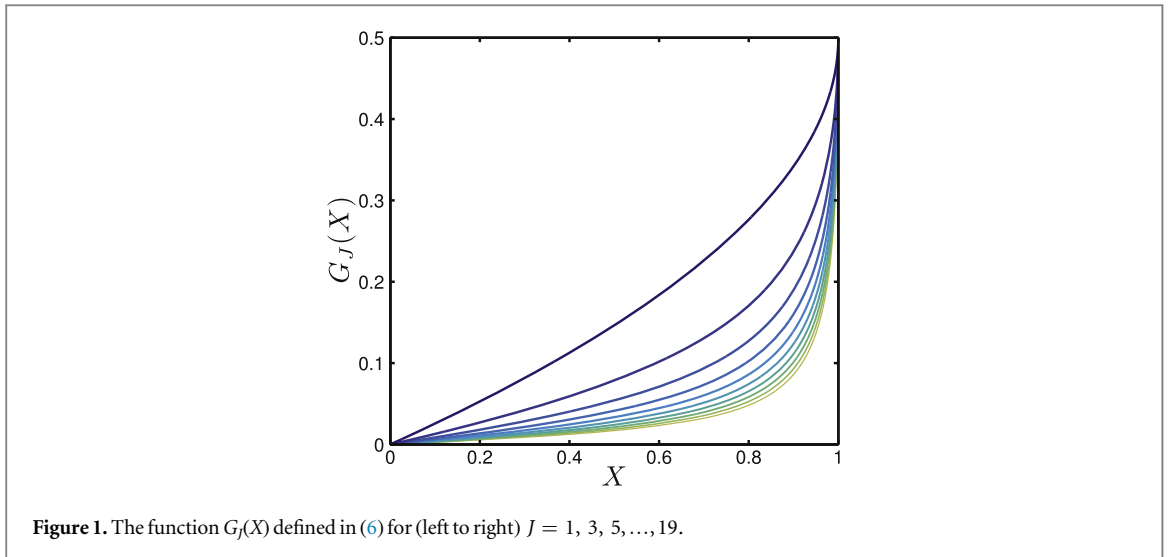
$$H_\lambda = L_x^2 - \lambda L_z \tag{10}$$

for a sufficiently wide interval of λ. Note that the ground states of (10) are the extreme spin-squeezed states studied in [18]. Then, the points of the curve $F_J(X)$ are obtained as $X = \frac{1}{J}\langle L_z\rangle_{\phi_\lambda}$ and $F_J(X) = \frac{1}{J}\langle L_x^2\rangle_{\phi_\lambda}$. Note that the method takes into account that the state minimizing $(\Delta J_x)^2$ for a given $\langle L_z\rangle$ has $\langle L_x\rangle = 0$, which is a property numerically observed to be true for integer J [18]. The algorithm can be extended to half-integer J's by adding a Lagrange multiplier term $\lambda_2 L_x$ that constraints $\langle L_x\rangle$ to some value, the details can be found in appendix A. In practice, $F_J(X)$ is computed typically for an integer J only, which makes it possible to detect $(k+1)$-particle entanglement for any k for an integer j and for an even k for a half-integer j. In the latter case, it is not a large restriction to consider only even k, since the entanglement depth in cold atom experiments can be quite large [7–9].

In a similar fashion, we can also obtain the curve for $G_J(X)$ defined in (6). The points of the curve are given as $X = \frac{1}{J^2}\langle L_z\rangle_{\phi_\lambda}^2$ and $G_J(X) = \frac{1}{J}\langle L_x^2\rangle_{\phi_\lambda}$. In figure 1, we drew $G_J(X)$ for various values of J. Based on these, the boundary for k-producible states in the $(\langle J_y^2 + J_z^2\rangle, (\Delta J_x)^2)$-plane is given by

$$\langle J_y^2 + J_z^2\rangle_\lambda = \frac{N(N-k)j^2}{k^2j^2}\langle L_z\rangle_{\phi_\lambda}^2 + Nj(kj+1),$$

$$(\Delta J_x)_\lambda^2 = \frac{N}{k}(\Delta L_x)_{\phi_\lambda}^2. \tag{11}$$

In the numerical calculations, L_l are Hermitian matrices of size $(2kj+1) \times (2kj+1)$. Hence, it is possible to draw the boundaries for various levels of multipartite entanglement for kj reaching up to the thousands, and for an arbitrarily large N.

We mention that for $J = 1$ we have $G_1(X) = \frac{1}{2}(1 - \sqrt{1-X})$, i.e., the function on the right-hand side of the criteria can be obtained analytically. Substituting $F_1(X) = G_1(X^2)$ into (5), we can obtain an analytic 2-producibility condition for qubits and an analytic separability condition for qutrits. In figure 2, we plotted the curves for k-producible states for some examples with spin-$\frac{1}{2}$ and spin-1 particles. For higher J, the function $G_J(X)$ is not known analytically. Based on uncertainty relations of angular momentum operators, a lower bound on $G_J(X)$ for any J can be obtained as

Figure 1. The function $G_J(X)$ defined in (6) for (left to right) $J = 1, 3, 5, \ldots, 19$.

Figure 2. 20-producibility criteria for $N = 200$ qubits. (solid) The boundary given by (5). (dashed) Criterion (7), i.e., the tangent to the curve given by (5). (dotted) Criterion (22) given in [39]. (inset) Curves for k-producibility for $N = 20$ spin-1 particles, for (left to right) $k = 1, 5, 9, 13, 17$.

$$\tilde{G}_J(X) = \frac{1}{2}[(J + 1) - JX - \sqrt{(J + 1 - JX)^2 - X}], \tag{12}$$

which is not tight for small J and small X, but becomes tight for large J and X close to 1 [18].

3. Nonlinear criterion

In this section, we present our first main result.

Observation 1. The inequality in (5) holds for all k-producible states of N spin-j particles. Thus, every state of N spin-j particles that violates (5) must be $(k + 1)$-entangled. The condition (5) can be used if $\langle J_y^2 + J_z^2 \rangle \geqslant Nj(kj + 1)$, while otherwise there is a k-producible quantum state for which $(\Delta J_x)^2 = 0$.

Proof. The key argument of the proof is that for pure k-producible states of N spin-j particles

$$\frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{Nj} \geqslant \sqrt{\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1)}{N(N - k)j^2}} \tag{13}$$

holds, which is proven in appendix B.1. Then, based on (13) and on the monotonicity of $F_J(X)$ in X, we have for pure k-producible states

$$F_J(\text{lhs}) \geqslant F_J(\text{rhs}). \tag{14}$$

Here, we used the notation lhs and rhs for the left-hand side and right-hand side of the relation (13), respectively. On the other hand, the Sørensen–Mølmer criterion (4) can be rewritten as

$$(\Delta J_x)^2 \geqslant Nj F_J(\text{lhs}). \tag{15}$$

From (14) and (15) follows that (5) holds for pure k-producible states.

Next, we will consider mixed states. In the formula (5) the argument of G is linear in the state. Then, our criterion (5) can be extended to mixed k-producible states via a convex hull of $G_J(X)$. However we can observe numerically that $G_J(X)$ is convex already by itself. The tightness of (5) is discussed in appendix B.2, while the convexity of $G_J(X)$ is considered in detail in appendix B.3. □

The criterion (5) is especially suited to detect states for which $\langle J_y^2 + J_z^2 \rangle$ is large and $(\Delta J_x)^2$ is small. A paradigmatic example for such a state is the unpolarized Dicke state in the x-basis

$$\rho_{\text{Dicke}} = |J = Nj, \, m_x = 0\rangle \langle J = Nj, \, m_x = 0|, \tag{16}$$

that satisfies $(\Delta J_x)^2 = 0$ and $\langle J_y^2 + J_z^2 \rangle = Nj(Nj + 1)$ and is detected as N-entangled. In fact, substituting these quantities in the criterion (5) the left-hand side is zero, while the right-hand side is positive for $k = N - 1$. Note also that the Dicke states violate maximally even the relation (13) for pure k-producible states[6].

4. Linear analytic criteria

In this section, we will derive the spin-squeezing parameters (7) and (8). Complementary to the approximation (12), our approach is based on a lower bound on $G_J(X)$ that is tighy for $X \approx 0$ and improves $\tilde{G}_J$ at small X by a factor of 2. For our derivation, we will compute the first terms of the Taylor expansion of $G_J(X)$ around $X = 0$. Using the convexity of $G_J(X)$, we will obtain the bound $G_J(X) \geqslant (G_J(0) + X G_J'(0))$, with $G_J(0) = 0$. In other words, we will compute the tangent to the k-producibility boundaries, near their intersection point with the horizontal axis.

In what follows, we present the details of the derivation. The expansion of $G_J(X)$ can be obtained by employing the perturbation series for H_λ in powers of the parameter $\lambda \ll 1$, taking advantage of the fact that $X = 0$ corresponds to $\lambda = 0$. The ground state of H_λ is then given by $|\phi_\lambda\rangle = |\phi^{(0)}\rangle + \lambda|\phi^{(1)}\rangle + O(\lambda^2)$[7], where $|\phi^{(0)}\rangle$ is the ground state of the unperturbed Hamiltonian $H^{(0)} = L_x^2$, i.e., the eigenstate of L_x with eigenvalue zero. As in usual perturbation theory, the first order term is obtained by imposing $\langle\phi^{(0)}|\phi^{(1)}\rangle = 0$ and results in $|\phi^{(1)}\rangle = \sum_{m \neq 0} c_m |E_m^{(0)}\rangle$, where $c_m = -\langle E_m^{(0)}|H^{(1)}|E_0^{(0)}\rangle/(E_m^{(0)} - E_0^{(0)})$ and $E_m^{(0)}$ are the energy levels of the unperturbed Hamiltonian. In our case, we obtain $|\phi^{(1)}\rangle = \sum_{m \neq 0} c_m |m\rangle_x$ with $c_m = -\langle m|_x L_z|0\rangle_x/m^2$, where $|m\rangle_x$ are the eigenstates of L_x with eigenvalue m. The expansion of the ground state explicitly is as follows

$$|\phi_\lambda\rangle = |0\rangle_x - i\lambda \frac{\sqrt{J(J+1)}}{2}(|1\rangle_x - |-1\rangle_x) + O(\lambda^2). \tag{17}$$

Based on (17), we obtain for the dependence of X and $G_J(X)$ on λ, respectively, $X(\lambda) = \frac{1}{J^2}\langle L_z\rangle_{\phi_\lambda}^2 \approx \lambda^2(J+1)^2$ and $G_J(X(\lambda)) = \frac{1}{J}\langle L_x^2\rangle_{\phi_\lambda} \approx \frac{1}{2}\lambda^2(J+1)$. Hence, we arrive at

$$G_J(X) \geqslant \frac{X}{2(J+1)}, \tag{18}$$

by employing the chain rule for $\frac{dG_J(X(\lambda))}{dX} = \frac{dG_J(X(\lambda))}{d\lambda}\Big/\frac{dX(\lambda)}{d\lambda}$ near $X = \lambda = 0$. Based on this, we can derive an analytic criterion that becomes tight close to the point $(\Delta J_x)^2 = 0$. Note that we could also use $\tilde{G}_J(X)$ defined in (12) instead of $G_J(X)$ for constructing our linear entanglement condition. However, taking the derivative of $\tilde{G}_J(X)$ one obtains $G_J(X) \geqslant \tilde{G}_J(X) \geqslant X\tilde{G}_J'(0) = \frac{X}{4(J+1)}$, which underestimates (18) by a factor of 2. Note that we computed the leading terms for the Taylor expansion of $G_J(X)$ analytically, while the function itself is known only numerically.

Observation 2. The criteria in (7) and (8) hold for all k-producible states of N spin-j particles such that J given in (2) is an integer number. Every state of N spin-j particles that violates one of the criteria must be $(k + 1)$-entangled, i.e., has an entanglement depth at least $k + 1$.

Proof. From (18) we can bound the criterion (5) from below with (7) by substituting $X = [\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1)]/[N(N - k)j^2]$. Analogously, by rewriting (4) in terms of G_J and using the bound (18) with $X = \langle J_z\rangle^2/N^2j^2$ we obtain (8). □

---

[6] We stress that (13) is not an entanglement criterion, since it does not hold for mixed k-producible states.

[7] $O(x)$ is the usual Landau symbol used to describe the asymptotic behavior of a quantity for small x.

In figure 2, we plot the criterion (7) as the tangent to the boundary of 20-producibility for $N = 200$ particles with spin $j = \frac{1}{2}$ in the $(\langle J_y^2 + J_z^2 \rangle, (\Delta J_x)^2)$-plane.

5. Comparison with similar criteria

Next, we compare our criteria with other similar entanglement conditions. First let us consider the Sørensen–Mølmer criterion (4).

Observation 3. Whenever the condition

$$\frac{(\Delta J_y)^2 + (\Delta J_z)^2}{Nj} > kj\left(1 - \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N^2 j^2}\right) + 1 \tag{19}$$

holds then our criterion (5) is strictly stronger than the Sørensen–Mølmer criterion (4).

Proof. (a) Since $F_J(X)$ is a monotonously increasing function of X, the inequality $F_J(X) \geqslant F_J(Y)$ holds if and only if $X \geqslant Y$. Hence, for comparing (4) and (5) it suffices to compare the arguments of the function F in the two conditions. It is then straightforward to prove that (5) implies (4) whenever (19) holds. Then, let us now present a family of states that are detected by (5), but not detected by (4). We consider states of the form

$$\rho_{\mathrm{Dicke},p} = (1 - p)\rho_{\mathrm{Dicke}} + p\frac{\mathbb{1}}{(2j + 1)^N}, \tag{20}$$

where the unpolarized Dicke state is given in (16). From the linear criterion (7) we obtain that if $p < \frac{3(N - k)j}{2j(j + 1)(kj + 1)(N - k) - 2(j + 1) + 3(Nj + 1)}$ then the state $\rho_{\mathrm{Dicke},p}$ is detected by (5). On the other hand, $\rho_{\mathrm{Dicke},p}$ is not detected by the Sørensen–Mølmer criterion (4), since $\langle J_l \rangle = 0$ for $l = x, y, z$ for this state for all p. $\qquad\square$

From observation 3, we can immediately see that our criterion (5) is much stronger than the original spin-squeezing criterion (4) for states close to Dicke states (16) since for such states $(\Delta J_y)^2 + (\Delta J_z)^2 \gg Nkj^2$. Here, we assumed that k is much smaller than N, which is consistent with experiments, where criterion (4) always detects an entanglement depth much smaller than N due to noise [7, 9].

Let us now study numerically how our criterion works for a relevant class of states. We consider spin-squeezed states of spin-$\frac{1}{2}$ particles obtained from ground states of the Hamiltonian

$$H_\mu = J_x^2 - \mu J_z, \tag{21}$$

for simplicity assuming an even particle number. The Dicke state (16) corresponds to $\mu = 0$, while the usual spin-squeezed states with a large spin polarization correspond to a large μ. For such states without noise, our criterion (5) is not stronger than (4).

Simple calculations show that if some small noise is present in the system then (5) detect an entanglement depth higher than the original criterion (4). First we consider spin-squeezed states for $N = 1000$ spin-$\frac{1}{2}$ particles, such that 10 particles are decohered into the fully mixed state. Such a noise is typical in cold atom experiments [43]. The results can be seen in figure 3. Our criterion (5) and the Sørensen–Mølmer criterion (4) detect the same entanglement depth for almost completely polarized spin-squeezed states. On the other hand, as the squeezing increases, our criterion detects a monotonically increasing entanglement depth, while the other criterion detects smaller and smaller multipartite entanglement. While we considered a noise affecting a few particles, still the detected entanglement depth is much smaller than N. Other types of noise, such as particle loss, small added while noise, or noise effects modeled considering the thermal states of (21) lead to a similar situation.

Next, we compare our criteria with another important condition that is designed to detect the depth of entanglement near unpolarized Dicke states (16). It is a linear criterion derived by Duan in [39], stating that

$$N(k + 2)(\Delta J_x)^2 \geqslant \langle J_y^2 + J_z^2 \rangle - \frac{N}{4}(k + 2) \tag{22}$$

holds for all k-producible states of N spin-$\frac{1}{2}$ particles. Any state that violates (22) is detected as $(k + 1)$-entangled. In this case, we can compare it with the linear criterion (7), specialized to qubit-systems, i.e., for $j = \frac{1}{2}$

$$\frac{(N - k)}{2}(k + 2)(\Delta J_x)^2 \geqslant \langle J_y^2 + J_z^2 \rangle - \frac{N}{4}(k + 2). \tag{23}$$
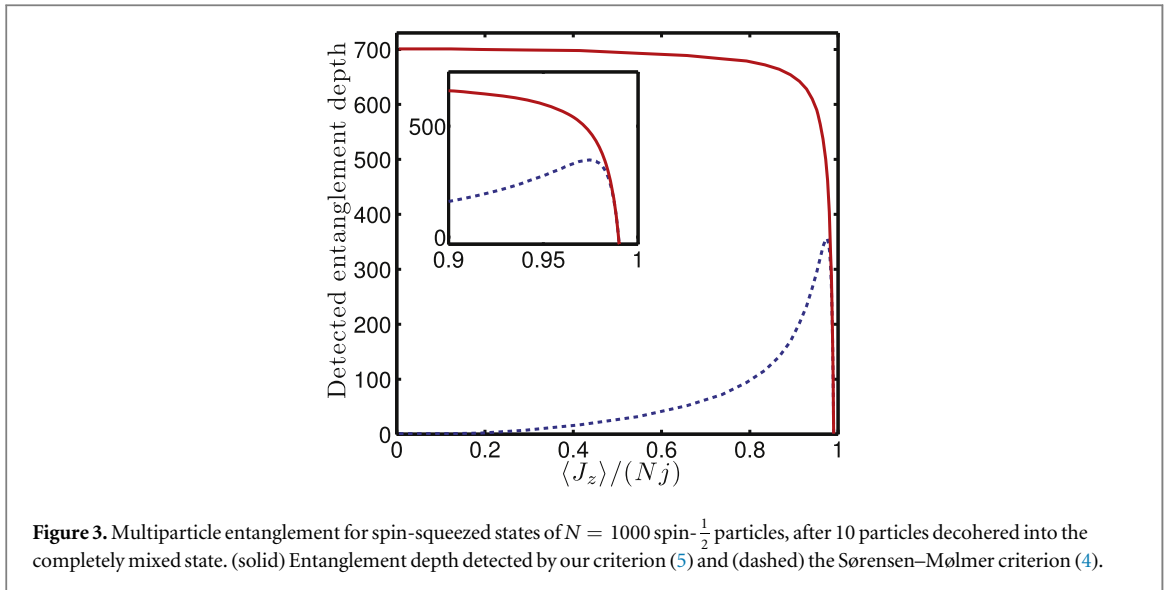
Figure 3. Multiparticle entanglement for spin-squeezed states of $N = 1000$ spin-$\frac{1}{2}$ particles, after 10 particles decohered into the completely mixed state. (solid) Entanglement depth detected by our criterion (5) and (dashed) the Sørensen–Mølmer criterion (4).

It is easy to see that a violation of (22) implies a violation of (23). Thus, our condition detects more states, which can be seen in figure 2.

Finally, we note that (5) with $j = 1/2$ is similar to the criterion for spin-1/2 particles used in the experiment described in [8]. A key difference is that in (5), in the denominator of the fraction, the term $N(N - k)j^2 = N(N - k)/4$ appears, while in the formula of [8] there is the term $N^2/4$. The difference between the two criteria is the largest when we examine highly entangled Dicke states or spin-squeezed states, and in the argument of $F_J(X)$ we have a value close to $X = 1$. In the vicinity of this point, the derivative of $F_J(X)$ is very large, hence some improvement in the argument of $F_J(X)$ makes the bound on the right-hand side of (5) significantly higher. As a consequence, the criterion (5) can be used to detect the noisy Dicke states of many particles even in $k \sim N$ case, while the criterion of [8] can be used only when $k \ll N$, and it does not detect the Dicke state as N-entangled.

6. Extension to fluctuating number of particles

For macroscopic ensembles of particles, e.g., for $N \sim 10^6$, the total particle number is not under perfect control. In this section, we will generalize our entanglement criteria to such a situation. The quantum state of a large particle ensemble with a fluctuating particle number is given as

$$\rho = \sum_N Q_N \rho_N, \tag{24}$$

where ρ_N are the density matrices corresponding to a subspace with a particle number N and Q_N are probabilities. We also have to consider collective spin operators defined as $J_l = \sum_N J_{l,N}$ for $l = x$, y, z, where $J_{l,N}$, act on the subspace with N particles. In principle, one could evaluate an entanglement condition, e.g., (4) for one of the fixed-N subspaces. If ρ_N has an entanglement depth k for some N, then the state ρ has also at least an entanglement depth k. However, in practice, we would not have sufficient statistics to evaluate our entanglement criteria for some fixed N. This issue has been studied by Hyllus *et al* [19], who generalized the definition of entanglement depth to the case of a fluctuating number of particles. They also showed how spin-squeezing criteria can be used in this case such that all the collected statistics is used, not only data for a given particle number N. For instance, (4) can be transformed to [19]

$$(\Delta J_x)^2 \geqslant \langle N \rangle j F_j \left(\frac{\langle J_z \rangle}{\langle N \rangle j} \right). \tag{25}$$

Here, (25) could be obtained from (4) simply with the substitution $N \to \langle N \rangle$.

Using methods similar to the ones in [19], we will now obtain the criterion (4) for fluctuating particle numbers.

Observation 4. All k-producible states with a fluctuating particle number must satisfy the following inequality

$$(\Delta J_x)^2 \geqslant \langle N \rangle j G_J\left(\frac{\langle W \rangle}{\langle N \rangle j}\right), \tag{26}$$

where we define the operator

$$W = \sum_N (Nj - kj)^{-1}[J_{y,N}^2 + J_{z,N}^2 - Nj(kj + 1)\mathbb{1}_N], \tag{27}$$

$\langle W \rangle \geqslant 0$ is required, and J is the total spin of a k-particle group given in (2).

Proof. We have to start from a state of the form (24). Due to the concavity of the variance, the variance of the mixed state can be bounded from below with the variances of ϱ_N as $(\Delta J_k)^2 \geqslant \sum_N Q_N (\Delta J_{k,N})^2$. Moreover, since $G_J(X)$ is convex in X, it has to satisfy Jensen's inequality. Thus,

$$\frac{1}{\sum_N Q_N N} \sum_N Q_N N G_J(X_N) \geqslant G_J\left(\frac{\sum_N Q_N N X_N}{\sum_N Q_N N}\right) \tag{28}$$

with $X_N = \frac{\langle W \rangle_{\rho_N}}{N}$ and $\langle N \rangle = \sum_N Q_N N$ must hold. Based on these, the statement of the observation follows. □

Note that the operator W defined in (27) is simply a sum of $J_{y,N}^2 + J_{z,N}^2$ over all fixed-N subspaces, normalized with $(Nj - kj)$. Thus, to apply our condition in experiments with fluctuating number of particles, one needs to measure the spin operators and the particle number jointly at each shot, and then average over an ensemble.

Finally, let us consider how to apply the ideas above for the spin-squeezing parameters defined in this paper. The parameter (8) can be extended to fluctuating particle numbers simply by replacing N with $\langle N \rangle$. Similarly, for the parameter (7), we have to replace $\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1)}{(N - k)j}$ by $\langle W \rangle$.

7. Conclusions

We derived a set of criteria to determine the depth of entanglement of spin-squeezed states and unpolarized Dicke states, extending and completing the results of [8, 18]. These generalized spin-squeezing conditions are valid even for an ensemble of spin-j particles with $j > \frac{1}{2}$, which is very useful, since most experiments are carried out with particles with a higher spin, e.g., with spin-1 ^{87}Rb atoms.

Since theory is mostly available for the spin-$\frac{1}{2}$ case, pseudo spin-$\frac{1}{2}$ particles are created artificially such that only two of the levels are populated. While the spin-squeezing approach to entanglement detection is already widely used in such systems [7, 8, 20–26, 33, 38], our criteria make it possible to study spin-squeezing in fundamentally new experiments. A clear advantage of using the physical spin is that it is typically much easier to manipulate than the pseudo spin-$\frac{1}{2}$ particles [33]. In future, it would be interesting to clarify the relation between generalized spin squeezing and metrological usefulness [44–49], and also compare our results with the complete set of spin-squeezing criteria of [50], which contain one additional collective observable, related to single-spin average squeezing.

Acknowledgments

Appendix A. Computing $F_J(X)$ and $G_J(X)$ for half-integer spin

For half-integer spins, we have to calculate $F_J(X)$ numerically as follows. We consider the Hamiltonian [18]

$$H_{\lambda,\lambda_2} = L_x^2 - \lambda L_z - \lambda_2 L_x, \tag{A.1}$$

and denote its ground state by $\phi_{\lambda, \lambda_2}$. Then, $F_J(X)$ can be obtained as

$$F_J(X) = \min_{\lambda, \lambda_2 : \frac{1}{J}\langle L_z \rangle = X} (\Delta J_x)^2_{\psi_{\lambda, \lambda_2}}, \tag{A.2}$$

which is a two-parameter optimization with the constraint $\frac{1}{J}\langle L_z \rangle = X$.

Appendix B. Details of the proof of observation 1

B.1. Proof of (13)
To prove (13), let us consider the expression $(\Delta J_y)^2 + (\Delta J_z)^2$ on pure k-producible states (9). Due to the additivity of the variance for tensor products

$$\begin{aligned}
(\Delta J_y)^2 + (\Delta J_z)^2 &= \sum_l [(\Delta j_y^{(l)})^2 + (\Delta j_z^{(l)})^2] \\
&\leqslant \sum_l [k_l j(k_l j + 1) - \langle (j_x^{(l)})^2 \rangle - \langle j_y^{(l)} \rangle^2 - \langle j_z^{(l)} \rangle^2]
\end{aligned} \tag{B.1}$$

holds, where the superscript (l) indicates the lth group, that is composed of k_l particles. The inequality (B.1) is saturated by all quantum states for which $\langle (j_x^{(l)})^2 + (j_y^{(l)})^2 + (j_z^{(l)})^2 \rangle$ is maximal, i.e., equal to $k_l j(k_l j + 1)$, for all l.

For simplifying our expression, we neglect the non-negative quantity

$$\mathcal{X} := \sum_l \langle (j_x^{(l)})^2 \rangle, \tag{B.2}$$

and after some rearrangement of the terms in (B.1) we arrive at

$$\begin{aligned}
\langle J_y^2 + J_z^2 \rangle &\leqslant \langle J_y \rangle^2 + \langle J_z \rangle^2 \\
&+ \sum_l k_l j \left[(k_l j + 1) - k_l j \frac{(\langle j_y^{(l)} \rangle^2 + \langle j_z^{(l)} \rangle^2)}{k_l^2 j^2} \right].
\end{aligned} \tag{B.3}$$

From (B.3), we can obtain a simpler bound as

$$\begin{aligned}
\langle J_y^2 + J_z^2 \rangle &\leqslant \langle J_y \rangle^2 + \langle J_z \rangle^2 + Nj \\
&+ \sum_l k_l j \left[kj \left(1 - \frac{\langle j_y^{(l)} \rangle^2 + \langle j_z^{(l)} \rangle^2}{k_l^2 j^2} \right) \right],
\end{aligned} \tag{B.4}$$

due to the fact that $k_l \leqslant k$, $\sum_l k_l = N$, and that the expression inside the round brackets in (B.4) is positive. Furthermore, using Jensen's inequality in the form

$$-\sum_l k_l f_l^2 \leqslant -\frac{1}{N} \left(\sum_l k_l f_l \right)^2, \quad \sum_l k_l = N, \tag{B.5}$$

with $f_l = \frac{\langle j_m^{(l)} \rangle}{k_l}$ for $m = x, y, z$ we obtain

$$\langle J_y^2 + J_z^2 \rangle - Nj(kj + 1) \leqslant \left(1 - \frac{k}{N} \right) (\langle J_y \rangle^2 + \langle J_z \rangle^2). \tag{B.6}$$
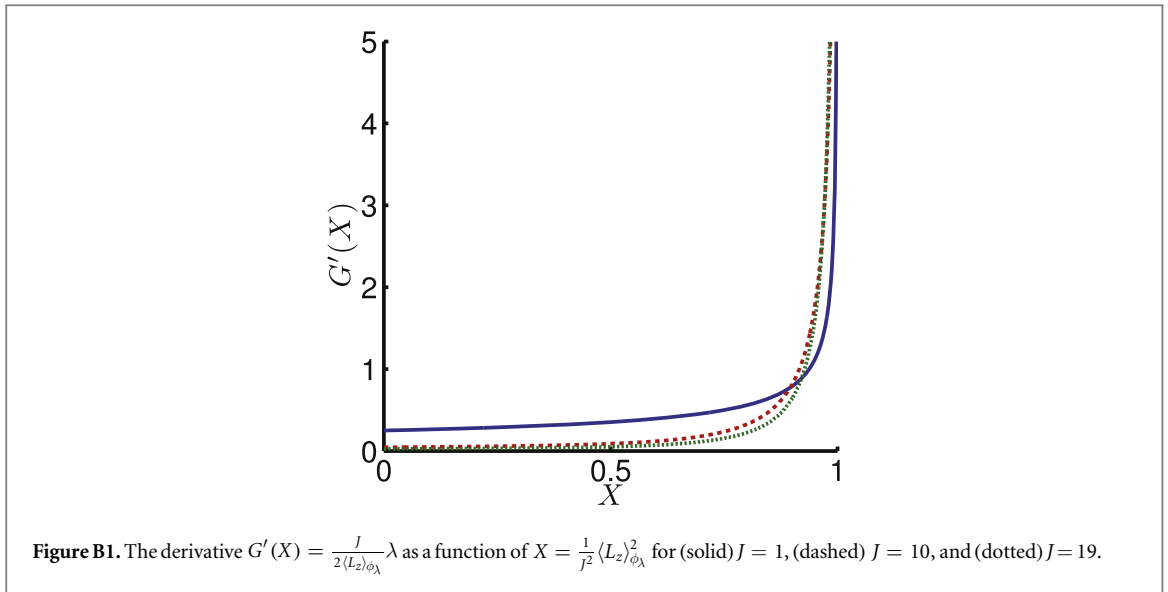
Hence, we proved (13).

B.2. Tightness of (13) and (5)
We will now examine, how the relation (13) would look for pure k-producible states (9) without neglecting $\mathcal{X}$ defined in (B.2). Simply, $\langle J_y^2 + J_z^2 \rangle$ would be substituted by $\langle J_y^2 + J_z^2 \rangle + \mathcal{X}$. With a derivation similar to the one in appendix B.1, it can be shown that such a condition would be saturated by all quantum states of the form $|\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle$, if $|\psi\rangle$ are k-qubit states and $\langle j_x^2 + j_y^2 + j_z^2 \rangle_\psi$ is maximal, i.e., it is $kj(kj + 1)$. (Here we assumed that $\mathcal{X}$ is defined such that all particle groups contain k particles, i.e, $k_l = k$ for all l.)

Let us now see how large $\mathcal{X}$ is for relevant states. For the state fully polarized in the z-direction, we have

$$\mathcal{X} = \sum_l (\Delta j_x^{(l)})^2 = Nj^2/2, \tag{B.7}$$

where we used the fact that $\langle j^{(l)} \rangle = 0$ for such a state. Let us consider now the ground states of the Hamiltonian (21) for a given parameter μ. Such states include usual spin-squeezed states, as well as Dicke states (16). For any μ,

Figure B1. The derivative $G'(X) = \frac{J}{2\langle L_z \rangle_{\phi_\lambda}}\lambda$ as a function of $X = \frac{1}{J^2}\langle L_z \rangle^2_{\phi_\lambda}$ for (solid) $J = 1$, (dashed) $J = 10$, and (dotted) $J = 19$.

$$\mathcal{X} < Nj^2/2 \tag{B.8}$$

holds, since for such states the variance of the x-components of the collective angular momentum is squeezed below that of the completely polarized state for any particle group. Note that the upper bound in (B.8) does not grow with k.

Let us now consider the other relevant quantity, $\langle J_y^2 + J_z^2 \rangle$. For the state fully polarized in the x-direction, we have $\langle J_y^2 + J_z^2 \rangle = Nj(Nj + 1/2)$. For the Dicke state (16), $\langle J_y^2 + J_z^2 \rangle = Nj(Nj + 1)$. For ground states of (21) for $\mu > 0$, $\langle J_y^2 + J_z^2 \rangle$ is in between these two values. This can be seen noticing that $\langle J_x^2 + J_y^2 + J_z^2 \rangle = Nj(Nj + 1)$ for these states.

Based on the previous discusion, it is clear that $\mathcal{X} \ll \langle J_y^2 + J_z^2 \rangle$ holds for large N. Hence, in practical cases the relation (13) is very close to being tight. Consequently, the criterion (5) also provides a tight bound on $(\Delta J_x)^2$ based on $\langle J_y^2 + J_z^2 \rangle$ in the large N limit.

B.3. Properties of F_J and G_J

The functions $F_J(X)$ can be obtained from the optimal states ρ for the problem defined in (3), i.e., the states that minimize $(\Delta L_x)^2$ for a given $\langle L_z \rangle$. As discussed in section 2, for an integer spin J, such states are the ground states of (10), where λ is a parameter. They have $\langle L_x \rangle = 0$ [18]. Thus, $F_J(X)$ yields the minimal $\langle J_x^2 \rangle$ for a given value of $\langle J_z \rangle$. Since the set of physical states is convex, the set of points in the ($\langle J_z \rangle$, $\langle J_x^2 \rangle$)-space corresponding to physical states is also convex. Hence, $F_J(X)$ is also a convex function and in particular its derivative $\lambda(X)$ is monotonously increasing with X. Note that in [18] a different proof was presented for this fact. In principle, the derivative $F_J'(X)$ can be computed by numerical derivation of $F_J(X)$. However, it is much simpler to obtain $F_J'(X)$ for some range of X by plotting $\left(\frac{1}{J}\langle L_z \rangle_{\phi_\lambda}, \lambda\right)$ for some range of λ [18]. In other words, for $X = \frac{1}{J}\langle L_z \rangle_{\phi_\lambda}$ the derivative is $F_J'(X) = \lambda$.

To show that also $G_J(X)$ is convex we observe that $G_J'(X) = \frac{1}{2\sqrt{X}}F_J'(\sqrt{X})$ is a monotonously increasing function of X. We evaluate numerically the derivative $G_J'(X)$ by plotting $\left(\frac{1}{J^2}\langle L_z \rangle^2_{\phi_\lambda}, \frac{J}{2\langle L_z \rangle_{\phi_\lambda}}\lambda\right)$ for a wide range of λ, see figure B1, and see explicitly its monotonicity.

More generally, one can check whether or not $F_J(X^{\frac{1}{\alpha}})$ is convex for any exponent α. It can then be observed numerically (not shown) that $F_J(X^{\frac{1}{\alpha}})$ is not convex for any $\alpha > 2$.

So far we discussed the case of integer spin. For half-integer spin, the ideas mentioned before cannot be used. Then, the derivative of G_J can be obtained via the numerical derivation of $F_J(X)$. Based on numerics, we can make the same statements about the convexity of $G_J(X)$ and $F_J(X^{\frac{1}{\alpha}})$ as for the case of an integer spin.

References

[1] Schwemmer C, Tóth G, Niggebaum A, Moroder T, Gross D, Gühne O and Weinfurter H 2014 *Phys. Rev. Lett.* **113** 040503
[2] Gao W B, Lu C Y, Yao X C, Xu P, Gühne O, Goebel A, Chen Y A, Peng C Z, Chen Z B and Pan J W 2010 *Nat. Phys.* **6** 331–5
[3] Leibfried D, Barrett M, Schaetz T, Britton J, Chiaverini J, Itano W, Jost J, Langer C and Wineland D 2004 *Science* **304** 1476–8
[4] Huang Y F, Liu B H, Peng L, Li Y H, Li L, Li C F and Guo G C 2011 *Nat. Commun.* **2** 546
[5] Appel J, Windpassinger P J, Oblak D, Hoff U B, Kjærgaard N and Polzik E S 2009 *Proc. Natl Acad. Sci.* **106** 10960–5

[6] Sewell R J, Koschorreck M, Napolitano M, Dubost B, Behbood N and Mitchell M W 2012 *Phys. Rev. Lett.* **109** 253605
[7] Gross C, Zibold T, Nicklas E, Esteve J and Oberthaler M K 2010 *Nature* **464** 1165–9
[8] Lücke B, Peise J, Vitagliano G, Arlt J, Santos L, Tóth G and Klempt C 2014 *Phys. Rev. Lett.* **112** 155304
[9] Hosten O, Engelsen N J, Krishnakumar R and Kasevich M A 2016 *Nature* **529** 505–8
[10] McConnell R, Zhang H, Hu J, Ćuk S and Vuletić V 2015 *Nature* **519** 439–42
[11] Haas F, Volz J, Gehr R, Reichel J and Esteve J 2014 *Science* **344** 180–3
[12] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865–942
[13] Gühne O and Tóth G 2009 *Phys. Rep.* **474** 1–75
[14] Pezzé L and Smerzi A 2009 *Phys. Rev. Lett.* **102** 100401
[15] Sørensen A, Duan L M, Cirac J and Zoller P 2001 *Nature* **409** 63–6
[16] Kitagawa M and Ueda M 1993 *Phys. Rev.* A **47** 5138–43
[17] Wineland D J, Bollinger J J, Itano W M and Heinzen D J 1994 *Phys. Rev.* A **50** 67–88
[18] Sørensen A S and Mølmer K 2001 *Phys. Rev. Lett.* **86** 4431–4
[19] Hyllus P, Pezzé L, Smerzi A and Tóth G 2012 *Phys. Rev.* A **86** 012337
[20] Hald J, Sørensen J L, Schori C and Polzik E S 1999 *Phys. Rev. Lett.* **83** 1319–22
[21] Fernholz T, Krauter H, Jensen K, Sherson J F, Sørensen A S and Polzik E S 2008 *Phys. Rev. Lett.* **101** 073601
[22] Orzel C, Tuchman A K, Fenselau M L, Yasuda M and Kasevich M A 2001 *Science* **291** 2386–9
[23] Riedel M F, Böhi P, Li Y, Hänsch T W, Sinatra A and Treutlein P 2010 *Nature* **464** 1170–3
[24] Esteve J, Gross C, Weller A, Giovanazzi S and Oberthaler M 2008 *Nature* **455** 1216–9
[25] Bohnet J G, Cox K C, Norcia M A, Weiner J M, Chen Z and Thompson J K 2014 *Nat. Photon.* **8** 731–6
[26] Cox K C, Greve G P, Weiner J M and Thompson J K 2016 *Phys. Rev. Lett.* **116** 093602
[27] Meyer V, Rowe M A, Kielpinski D, Sackett C A, Itano W M, Monroe C and Wineland D J 2001 *Phys. Rev. Lett.* **86** 5870–3
[28] Auccaise R, Araujo-Ferreira A G, Sarthour R S, Oliveira I S, Bonagamba T J and Roditi I 2015 *Phys. Rev. Lett.* **114** 043604
[29] Mitchell M W and Beduini F A 2014 *New J. Phys.* **16** 073027
[30] Tóth G, Knapp C, Gühne O and Briegel H J 2007 *Phys. Rev. Lett.* **99** 250405
[31] He Q Y, Peng S G, Drummond P D and Reid M D 2011 *Phys. Rev.* A **84** 022107
[32] Reis M S, Soriano S, dos Santos A M, Sales B C, Soares-Pinto D O and Brandão P 2012 *Europhys. Lett.* **100** 50001
[33] Behbood N, Ciurana F M, Colangelo G, Napolitano M, Tóth G, Sewell R J and Mitchell M W 2014 *Phys. Rev. Lett.* **113** 093601
[34] Vitagliano G, Apellaniz I, Egusquiza I L and Tóth G 2014 *Phys. Rev.* A **89** 032307
[35] Wieczorek W, Krischek R, Kiesel N, Michelberger P, Tóth G and Weinfurter H 2009 *Phys. Rev. Lett.* **103** 020504
[36] Prevedel R, Cronenberg G, Tame M S, Paternostro M, Walther P, Kim M S and Zeilinger A 2009 *Phys. Rev. Lett.* **103** 020503
[37] Lücke B *et al* 2011 *Science* **334** 773–6
[38] Hamley C, Gerving C, Hoang T, Bookjans E and Chapman M 2012 *Nat. Phys.* **8** 305–8
[39] Duan L M 2011 *Phys. Rev. Lett.* **107** 180502
[40] Zhang Z and Duan L M 2013 *Phys. Rev. Lett.* **111** 180401
[41] Hoang T M, Anquez M, Boguslawski M J, Bharath H M, Robbins B A and Chapman M S 2016 *Proc. Natl Acad. Sci.* **113** 9475
[42] Gühne O, Tóth G and Briegel H J 2005 *New J. Phys.* **7** 229
[43] de Echaniz S R, Mitchell M W, Kubasik M, Koschorreck M, Crepaz H, Eschner J and Polzik E S 2005 *J. Opt.* B **7** S548
[44] Hyllus P, Laskowski W, Krischek R, Schwemmer C, Wieczorek W, Weinfurter H, Pezzé L and Smerzi A 2012 *Phys. Rev.* A **85** 022321
[45] Tóth G 2012 *Phys. Rev.* A **85** 022322
[46] Zhang Z and Duan L M 2014 *New J. Phys.* **16** 103037
[47] Tóth G and Apellaniz I 2014 *J. Phys. A: Math. Theor.* **47** 424006
[48] Apellaniz I, Lücke B, Peise J, Klempt C and Tóth G 2015 *New J. Phys.* **17** 083027
[49] Apellaniz I, Kleinmann M, Gühne O and Tóth G 2015 arXiv:1511.05203
[50] Vitagliano G, Hyllus P, Egusquiza I L and Tóth G 2011 *Phys. Rev. Lett.* **107** 240502