Practical methods for witnessing genuine multi-qubit entanglement in the vicinity of symmetric states

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Abstract. We present general numerical methods to construct witness operators for entanglement detection and estimation of the fidelity. Our methods are applied to detecting entanglement in the vicinity of a six-qubit Dicke state with three excitations and also to further entangled symmetric states. All our witnesses are designed to keep the measurement effort small. We also present general results on the efficient local decomposition of permutationally invariant operators, which makes it possible to measure projectors to symmetric states efficiently.

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1. Introduction

Entanglement plays a central role in quantum mechanics and in quantum information processing applications [1]. Moreover, it is also the main goal in today's quantum physics experiments aiming to create various quantum states [2]. For example, entanglement has been realized with photonic systems using parametric down-conversion and conditional detection [3]–[9], with trapped cold ions [10]–[12], in cold atomic ensembles [13], in cold atoms in optical lattices [14] and in diamond between the electron and nuclear spins [15]. These experiments aimed at creating entangled states. Entanglement makes it possible for some quantum algorithms (e.g. prime factoring, searching in a database) to outperform their classical counterparts. Entangled particles are needed for quantum teleportation and other quantum communication protocols. Moreover, the creation of large entangled states might lead to new insights about how a classical macroworld emerges from a quantum microworld.

In a multi-qubit experiment, typically the full density matrix is not known, and only few measurements can be made, yet one would still like to ensure that the prepared state is entangled. One possibility is applying entanglement witnesses [16, 17]. These are observables that have a positive expectation value for separable states, while for some entangled states their expectation value is negative. Since these witness operators are multi-qubit operators, they typically cannot be measured directly and must be decomposed into the sum of locally measurable operators, which are just products of single-qubit operators [4, 18, 19].

For many quantum states, like the Greenberger–Horne–Zeilinger (GHZ, [20]) states and the cluster states [21] such a decomposition of projector-based witness operators seems to be

very difficult: the number of terms in a decomposition to a sum of products of Pauli matrices increases rapidly with the number of qubits. However, practically useful entanglement witnesses with two measurement settings can be constructed for such states [5, 22]. It also turned out that there are decompositions of the projector for GHZ and W states in which the increase with the number of qubits is linear [23].

However, optimal decomposition of an operator is a very difficult, unsolved problem. Moreover, in general, it is still a difficult task to construct efficient entanglement witnesses for a given quantum state. For that, typically we need to obtain the maximum of some operators for product states. In most of the cases, we would like to detect genuine multipartite entanglement. For that, we need to obtain the maximum of these operators for biseparable states, which is again a very hard problem.

In this paper, our goal is to design witnesses that make it possible to detect genuine multipartite entanglement with few measurements, and also to estimate the fidelity of an experimentally prepared state with respect to the target state. Here three strategies are applied to find an experimentally realizable witness. (i) The first strategy is based on measuring the projector-based witness

$$\mathcal{W}^{(P)} = \text{const.} \cdot \mathbb{1} - |\Psi\rangle\langle\Psi| \tag{1}$$

for the detection of genuine multipartite entanglement. $|\Psi\rangle$ is the target state of the experiment. For reducing experimental effort, the aim is to find an efficient decomposition of the projector. (ii) The second strategy is to find a witness that needs fewer measurements than the projector witness, but the price for that might be a lower robustness against noise. The search for such a witness can be simplified if we look for a witness $\mathcal{W}$ such that

$$\mathcal{W} - \alpha \mathcal{W}^{(P)} \geqslant 0 \tag{2}$$

for some $\alpha > 0$. Such a witness is guaranteed to detect genuine multi-qubit entanglement. The advantage of this approach is that the expectation value of $\mathcal{W}$ can be used to find a lower bound on the fidelity. (iii) The third strategy is to find a witness independent from the projector witness. In this case, one has to find an easily measurable operator whose expectation value takes its maximum for the target state. Then, one has to find the maximum of this operator for biseparable states. Any state that has an operator expectation value larger than that is genuine multipartite entangled.

For the optimization of entanglement witnesses for small experimental effort and large robustness to noise, we use semidefinite programming [24]–[29]. Our methods can efficiently be used for multi-qubit systems with up to about 10 qubits. This is important, since there are many situations where semidefinite programming could help theoretically, but in practice the calculations cannot be carried out even for systems of modest size.

We use our methods to design witnesses detecting entanglement in the vicinity of symmetric Dicke states. An N-qubit symmetric Dicke state with m excitations is defined as [30, 31]

$$|D_N^{(m)}\rangle := \binom{N}{m}^{-1/2} \sum_k \mathcal{P}_k(|1_1, 1_2, \ldots, 1_m, 0_{m+1}, \ldots, 0_N\rangle), \tag{3}$$

where $\sum_k \mathcal{P}_k(.)$ denotes summation over all distinct permutations of the spins. $|D_N^{(1)}\rangle$ is the well-known N-qubit W state. The witnesses we will introduce in the following have already been used in the photonic experiment described in [32], aiming to observe a $|D_6^{(3)}\rangle$

state [32][8]. We show that genuine multi-qubit entanglement can be detected and the fidelity with respect to the above highly entangled state can efficiently be estimated with two and three measurement settings, respectively. As a byproduct, we will also derive an upper bound for the number of settings needed to measure any permutationally invariant operator. We show that such operators can be efficiently measured even for large systems.

The structure of our paper is as follows. In section 2, we present the basic methods for constructing witnesses. In section 3, we use these methods for constructing witnesses to detect entanglement in the vicinity of a six-qubit symmetric Dicke state with three excitations. In section 4, we present witnesses for states obtained from the above state by measuring some of the qubits. In appendix A, we summarize the tasks that can be solved by semidefinite programming, when looking for suitable entanglement witnesses. In appendix B, we summarize some of the relevant numerical routines of the QUBIT4MATLAB 3.0 program package [34]. In appendix C, we present entanglement conditions for systems with 5–10 qubits that will be relevant in future experiments.

2. Basic definitions and general methods

A multi-qubit quantum state is entangled if it cannot be written as a convex combination of product states. However, in a multi-qubit experiment we would like to detect genuine multi-qubit entanglement [35]: the presence of such entanglement indicates that all the qubits are entangled with each other, not only some of them. We will now need the following definitions:

Definition 1. A pure multi-qubit quantum state is called **biseparable** if it can be written as the tensor product of two, possibly entangled, multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle. \tag{4}$$

A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states. If a state is not biseparable then it is called **genuine multi-partite entangled**. In this paper, we will consider witness operators that detect genuine multipartite entanglement.

Definition 2. While an entanglement witness is an observable, typically it cannot be measured directly. This is because in most experiments only local measurements are possible. At each qubit k we are able to measure a single-qubit operator M_k, which we can do simultaneously at all the qubits. If we repeat such measurements, then we obtain the expectation values of $2^N - 1$ multi-qubit operators. For example, for $N = 3$ these are $M_1 \otimes \mathbb{1} \otimes \mathbb{1}$, $\mathbb{1} \otimes M_2 \otimes \mathbb{1}$, $\mathbb{1} \otimes \mathbb{1} \otimes M_3$, $M_1 \otimes M_2 \otimes \mathbb{1}$, $M_1 \otimes \mathbb{1} \otimes M_3$, $\mathbb{1} \otimes M_2 \otimes M_3$, $M_1 \otimes M_2 \otimes M_3$. The set of single-qubit operators measured is called the **measurement setting** [4] and it can be given as $\{M_1, M_2, M_3, \ldots, M_N\}$. When we consider an entanglement condition, it is important to know how many measurement settings are needed for its evaluation.

Definition 3. Many experiments aim at preparing some, typically pure quantum state ϱ. An entanglement witness is then designed to detect the entanglement of this state. However, in real experiments such a state is never produced perfectly, and the realized state is mixed with noise as given by the following formula:

$$\varrho_{\text{noisy}}(p_{\text{noise}}) = (1 - p_{\text{noise}})\varrho + p_{\text{noise}}\varrho_{\text{noise}}, \tag{5}$$

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[8] For another experiment aiming to observe a six-qubit Dicke state see Prevedel *et al* [33]. See also the related theoretical work of Campbell *et al* [33].

where p_noise is the ratio of noise and ϱ_noise is the noise. If we consider white noise then $\varrho_\text{noise} = \mathbb{1}/2^N$. The **noise tolerance** of a witness $\mathcal{W}$ is characterized by the largest p_noise for which we still have $\text{Tr}(\mathcal{W}\varrho_\text{noisy}) < 0$.

In this paper, we will consider three possibilities for detecting genuine multi-qubit entanglement, explained in the following subsections. Later, we will use these ideas to construct various entanglement witnesses.

2.1. Projector witness

A witness detecting genuine multi-qubit entanglement in the vicinity of a pure state $|\Psi\rangle$ can be constructed with the projector as

$$\mathcal{W}_\Psi^{(P)} := \lambda_\Psi^2 \mathbb{1} - |\Psi\rangle\langle\Psi|, \tag{6}$$

where λ is the maximum of the Schmidt coefficients for $|\Psi\rangle$, when all bipartitions are considered [4]. For the states considered in this paper, projector-based witnesses are given by [4, 12, 37]

$$\mathcal{W}_{D(N,N/2)}^{(P)} := \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|, \tag{7}$$

$$\mathcal{W}_{D(N,1)}^{(P)} := \frac{N-1}{N}\mathbb{1} - |D_N^{(1)}\rangle\langle D_N^{(1)}|. \tag{8}$$

These witnesses must be decomposed into the sum of locally measurable terms. For this decomposition, the following observations will turn out to be very important.

Observation 1. A permutationally invariant operator A can always be decomposed as [45]

$$A = \sum_n c_n a_n^{\otimes N}, \tag{9}$$

where a_n are single-qubit operators, and such a decomposition can be straightforwardly obtained.

Proof. Any permutationally invariant multi-qubit operator A can be decomposed as

$$A = \sum_n c_n \sum_k P_k(B_{n,1} \otimes B_{n,2} \otimes B_{n,3} \otimes \cdots \otimes B_{n,N})P_k, \tag{10}$$

where $B_{n,m}$ are single-qubit operators, c_n are constants, and P_k are the full set of operators permuting the qubits. For odd N, we can use the identity

$$\sum_k P_k(B_{n,1} \otimes B_{n,2} \otimes B_{n,3} \otimes \cdots \otimes B_{n,N})P_k$$
$$= 2^{-(N-1)} \sum_{\substack{s_1,s_2,\ldots=\pm 1, \\ s_1 s_2 s_3 \cdots s_N=+1}} (s_1 B_{n,1} + s_2 B_{n,2} + s_3 B_{n,3} + \cdots)^{\otimes N}. \tag{11}$$

Substituting (11) into (10), we obtain a decomposition of the form (9). Equation (11) can be proved by carrying out the summation and expanding the brackets. Due to the $s_1 s_2 s_3 \cdots s_N = +1$ condition, the coefficient of $B_{n,1} \otimes B_{n,2} \otimes B_{n,3} \otimes \cdots \otimes B_{n,N}$ is 1. The coefficient of terms like $B_{n,1} \otimes B_{n,1} \otimes B_{n,3} \otimes \cdots \otimes B_{n,N}$, that is, terms containing one of the variables more than once is zero. For even N, a similar proof can be carried out using[9]

$$\sum_k P_k (B_{n,1} \otimes B_{n,2} \otimes B_{n,3} \otimes \cdots \otimes B_{n,N}) P_k$$
$$= 2^{-(N-1)} \sum_{\substack{s_1,s_2,\ldots=\pm 1, \\ s_1 s_2 s_3 \cdots s_N = +1}} s_1 (B_{n,1} + s_2 B_{n,2} + s_3 B_{n,3} + \cdots)^{\otimes N}. \tag{12}$$

Next, we give two examples for the application of (11) and (12) for the decomposition of simple expressions

$$\sum_k P_k (\sigma_x \otimes \sigma_y) P_k = \frac{1}{2} \left\{ (\sigma_x + \sigma_y)^{\otimes 2} - (\sigma_x - \sigma_y)^{\otimes 2} \right\}, \tag{13}$$

$$\sum_k P_k (\sigma_x \otimes \sigma_y \otimes \sigma_z) P_k = \frac{1}{4} \left\{ (\sigma_x + \sigma_y + \sigma_z)^{\otimes 3} + (\sigma_x - \sigma_y - \sigma_z)^{\otimes 3} + (-\sigma_x - \sigma_y + \sigma_z)^{\otimes 3} \right.$$
$$\left. + (-\sigma_x + \sigma_y - \sigma_z)^{\otimes 3} \right\}, \tag{14}$$

where σ_k are the Pauli spin matrices. While the first example does not reduce the number of settings needed, the second example reduces the number of settings from 6 to 4. □

Next, we present a method to get efficient decompositions for permutationally invariant operators.

Observation 2. Any N-qubit permutationally invariant operator A can be measured with at most

$$\mathcal{L}_N = \frac{2}{3} N^3 + N^2 + \frac{4}{3} N \tag{15}$$

local measurement settings, using (11) and (12).

Proof. We have to decompose first A into the sum of Pauli group elements as

$$A = \sum_{i,j,m: \, i+j+m \leqslant N} c_{ijm} \sum_k P_k (\sigma_x^{\otimes i} \otimes \sigma_y^{\otimes j} \otimes \sigma_z^{\otimes m} \otimes \mathbb{1}^{\otimes (N-i-j-m)}) P_k, \tag{16}$$

where c_{ijm} are some constants. Then, such a decomposition can be transformed into another one of the form (9), using (11) and (12). All of the settings needed are of the form $\{a, a, a, \ldots, a\}$ where $a = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$, n_k are integer and $1 \leqslant \sum_k |n_k| \leqslant N$. Simple counting leads to an upper bound $\mathcal{L}_N$ for the number of settings given in (15). Here we considered that (n_x, n_y, n_z) and $(-n_x, -n_y, -n_z)$ describe the same setting. An even better bound can be obtained using that (n_x, n_y, n_z) and (cn_x, cn_y, cn_z) for some $c \neq 0$ represent the same setting. An algorithm based

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[9] A similar decomposition with continuous number of terms is of the form $\sum_k P_k (B_1 \otimes B_2 \otimes \ldots) P_k \propto \int_{\phi_k \in [0,2\pi]} [e^{i\phi_1} B_1 + e^{i\phi_2} B_2 + \cdots + e^{i\phi_{N-1}} B_{N-1} + e^{-i(\phi_1+\phi_2+\cdots+\phi_{N-1})} B_N]^{\otimes N} d\phi_1 d\phi_2 \cdots d\phi_{N-1}$. Such a construction has been used for the $N=2$ case in [48].

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on this leads to the bounds $\mathcal{L}'_N = 9, 25, 49, 97, 145, 241, 337, 481, 625$ for $N = 2, 3, \ldots, 10$ qubits, respectively.

For the projector $|D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|$, the decomposition to Pauli group elements contain only terms in which each Pauli matrix appears an even number of times. Hence, all of the settings needed are of the form $\{a, a, a, \ldots, a\}$ where $a = 2n_x\sigma_x + 2n_y\sigma_y + 2n_z\sigma_z$, n_k are integer and $1 \leqslant \sum_k |n_k| \leqslant N/2$. For this reason, $\mathcal{L}_{N/2}$ and $\mathcal{L}'_{N/2}$ are upper bounds for the number of settings needed to measure this operator.

Let us discuss the consequences of observations 1 and 2. They essentially state that the number of settings needed to measure a permutationally invariant operator scales only polynomially with the number of qubits. This is important since for operators that are not permutationally invariant, the scaling is known to be exponential [36]. Moreover, even if we can measure only correlation terms of the form $a^{\otimes N}$, we can measure any permutationally invariant operator. □

2.2. Witnesses based on the projector witness

We can construct witnesses that are easier to measure than the projector witness, but they are still based on the projector witness. We use the idea mentioned in the introduction. If $\mathcal{W}^{(P)}$ is the projector witness and (2) is fulfilled for some $\alpha > 0$, then $\mathcal{W}$ is also a witness. This is because $\mathcal{W}$ has a negative expectation value only for states for which $\mathcal{W}^{(P)}$ also has a negative expectation value. The advantage of obtaining witnesses this way is that we can have a lower bound on the fidelity from the expectation value of the witness as

$$\text{Tr}(\varrho|\Psi\rangle\langle\Psi|) \geqslant \lambda_\Psi^2 - \frac{1}{\alpha}\text{Tr}(\mathcal{W}\varrho). \tag{17}$$

We will look for such witnesses numerically, such that the noise tolerance of the witness be the largest possible. This search can be simplified by the following observation.

Observation 3. Since we would like to construct a witness detecting genuine multi-qubit entanglement in the vicinity of a permutationally invariant state, it is enough to consider witness operators that are also permutationally invariant.

Proof. Let us consider a witness operator that detects entanglement in the vicinity of a permutationally invariant state ϱ and its expectation value takes its minimum for ϱ. Then, based on (5), the witness $\mathcal{W}$ detects entanglement if

$$p_{\text{noise}} > \frac{\text{Tr}(W\varrho)}{\text{Tr}(W\varrho) - \text{Tr}(W\varrho_{\text{noise}})}. \tag{18}$$

For a permutationally invariant state ϱ, we have $\varrho = \frac{1}{N_P}\sum_k P_k\varrho P_k$, where N_P is the number of different permutation operators P_k. We assume that the same holds also for ϱ_{noise}. Let us define the permutationally invariant operator $\mathcal{W}' = \frac{1}{N_P}\sum_k P_k\mathcal{W}P_k$. The operator $\mathcal{W}'$ is non-negative on all biseparable states since

$$\inf_{\varrho\in\mathcal{B}}\text{Tr}(W\varrho) = \frac{1}{N_P}\sum_k \inf_{\varrho\in\mathcal{B}}\text{Tr}(\mathcal{W}P_k\varrho P_k) \leqslant \inf_{\varrho\in\mathcal{B}}\text{Tr}(\mathcal{W}'\varrho), \tag{19}$$

where $\mathcal{B}$ is the set of biseparable states. Hence, $\mathcal{W}'$ is a witness detecting genuine multipartite entanglement. Since we have $\text{Tr}(\mathcal{W}\varrho) = \text{Tr}(\mathcal{W}'\varrho)$, and $\text{Tr}(\mathcal{W}\varrho_{\text{noise}}) = \text{Tr}(\mathcal{W}'\varrho_{\text{noise}})$,

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the robustness to noise of $\mathcal{W}'$ is identical to that of $\mathcal{W}$. Hence, it is sufficient to look for witnesses that are permutationally invariant.

We will first consider measuring the $\{\sigma_x, \sigma_x, \sigma_x, \sigma_x, \sigma_x, \sigma_x\}$ and $\{\sigma_y, \sigma_y, \sigma_y, \sigma_y, \sigma_y, \sigma_y\}$ settings, where σ_l are the Pauli spin matrices. This we call the two-setting case. Then we will consider measuring also the $\{\sigma_z, \sigma_z, \sigma_z, \sigma_z, \sigma_z, \sigma_z\}$ setting, which we call the three-setting case. Due to observation 3, we consider only permutationally invariant witnesses. Such witnesses can be written as

$$\mathcal{W}(\alpha_0, \{\alpha_{ln}\}) := \alpha_0 \cdot \mathbb{1} + \sum_{l=x,y,z} \sum_{n=1}^{N} \alpha_{ln} \sum_k \mathcal{P}_k[\sigma_l^{\otimes n} \otimes \mathbb{1}^{\otimes(N-n)}], \tag{20}$$

where the summation is over all distinct permutations, and α_0 and α_{ln} are some constants. We will consider a simpler but equivalent formulation

$$\mathcal{W}(c_0, \{c_{ln}\}) := c_0 \cdot \mathbb{1} + \sum_{l=x,y,z} \sum_{n=1}^{N} c_{ln} J_l^n, \tag{21}$$

where c_0, c_{ln} are the coefficients of the linear combination defining the witness and J_l are the components of the total angular momentum given as

$$J_l = \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)}. \tag{22}$$

Here $\sigma_l^{(k)}$ denotes a Pauli spin matrix acting on qubit (k).

Finally, if we consider detecting entanglement in the vicinity of $|D_N^{(N/2)}\rangle$ states, then further simplifications can be made. For this state and also for the completely mixed state all odd moments of J_l have a zero expectation value. For any witness of the form (21), the maximum for biseparable states does not change if we flip the sign of c_{ln} for all odd n. Hence, following from an argument similar to the one in observation 3 concerning permutational symmetry, it is enough to consider only even powers of J_l in our witnesses. $\qquad\square$

2.3. Witnesses independent from the projector witness

In general, we can also design witnesses without any relation to the projector witness. We can use an easily measurable operator M to make a witness of the form

$$\mathcal{W} := c\mathbb{1} - M, \tag{23}$$

where c is some constant. To make sure that (23) is a witness for genuine multipartite entanglement, i.e. $\langle \mathcal{W} \rangle$ is positive on all biseparable states, we have to set c to

$$c = \max_{|\Psi\rangle \in \mathcal{B}} \langle M \rangle_{|\Psi\rangle}, \tag{24}$$

where $\mathcal{B}$ is the set of biseparable states. The optimization needed for (24) can be done analytically. For example, for the $|D_4^{(2)}\rangle$ state a witness has been presented that detects genuine four-qubit entanglement by measuring second moments of angular momentum operators [37]. However, analytical calculations become exceedingly difficult as the number of qubits increases.

The optimization can also be done numerically, but one cannot be sure that simple numerical optimization finds the global maximum. (See appendix B for a reference to such a MATLAB program.) Semidefinite programming is known to find the global optimum, but

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the optimization task (24) cannot be solved directly by semidefinite programming. Instead of looking for the maximum for biseparable states, using semidefinite programming, we can look for the maximum for states that have a positive partial transpose (PPT) [24, 38] (see appendices A and B). This way we can obtain

$$c' := \max_{I} \max_{\varrho \geqslant 0, \varrho^{T_I} \geqslant 0} \langle M \rangle_{\varrho}, \tag{25}$$

for which $c' \geqslant c$. The first maximization is over all bipartitions I. Thus, when putting c' into the place of c in (23), we obtain a witness that detects only genuine multipartite entanglement. In many cases simple numerics show that $c = c'$. In this case, our witnesses are optimal in the sense that some biseparable state gives a zero expectation value for these witnesses.

Finally, let us discuss how to find the operator M in (23) for a two- or a three-setting witness, in particular, for detecting entanglement in the vicinity of $|D_N^{(N/2)}\rangle$. Based on section 2.2, we have to look for an operator that contains only even powers of J_l. Hence, the general form of a two-setting witness with moments up to second order is

$$\mathcal{W}_{D(N,N/2)}^{(I2)} := c_{DN} - (J_x^2 + J_y^2), \tag{26}$$

where c_{DN} is a constant[10]. The coefficients of J_x^2 and J_y^2 could still be different, however, this would not lead to witnesses with a better robustness to noise.

For other symmetric Dicke states, based on similar arguments, a general form of a witness containing moments of J_l up to second order such that it takes its minimum for $|D_N^{(m)}\rangle$ is of the form

$$\mathcal{W}_{D(N,m)}^{(I3)} := c_q - (J_x^2 + J_y^2) + q(J_z - \langle J_z \rangle_{|D_N^{(m)}\rangle})^2, \tag{27}$$

where c_q and q are constants. For the witnesses described in this section, the optimization process is more time-consuming than for the witnesses related to the projector witness. Because of that we presented witnesses of the above type that are constructed only with the first and second moments of the angular momentum operators, and thus contain a few free parameters.

3. Witnesses for a six-qubit Dicke state with three excitations

In this section, we will consider entanglement detection close to a six-qubit symmetric Dicke state with three excitations, denoted as $|D_6^{(3)}\rangle$. There are several proposals for creating Dicke states in various physical systems [40]–[43].

3.1. Witnesses based on the projector witness

3.1.1. Two-setting witness. Let us consider the two-setting case and define first the optimization problem we want to solve. We would like to look for the witness $\mathcal{W}$ with the largest noise tolerance that fulfills the following requirements:

1. $\mathcal{W}$ is a linear combination of certain basis operators B_k, that is, $\mathcal{W} = \sum_k c_k B_k$,
2. $\mathcal{W} - \alpha \mathcal{W}_{D(6,3)}^{(P)} \geqslant 0$ with some $\alpha > 0$.

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[10] Witnesses for the state $|D_6^{(3)}\rangle$ are presented with the structure factor in [39]. In a sense, these witnesses are written with collective quantities, after a site-dependent phase shift is applied.

For the two-setting case we set $\{B_k\} = \{\mathbb{1}, J_x^2, J_y^2, J_x^4, J_y^4, J_x^6, J_y^6\}$. The second condition makes sure that $\mathcal{W}$ is also a witness detecting genuine multipartite entanglement.

Note that *any* optimization algorithm can be used for looking for $\mathcal{W}$. Even if we do not find the global optimum, that is, the witness with the largest possible robustness to white noise, $\mathcal{W}$ is still a witness detecting genuine multipartite entanglement. However, semidefinte programming can be used to find the global optimum (see appendix A). The two-setting witness obtained this way is

$$\mathcal{W}_{\mathrm{D}(6,3)}^{(P2)} := 7.75 \cdot \mathbb{1} - \tfrac{35}{18}(J_x^2 + J_y^2) + \tfrac{55}{72}(J_x^4 + J_y^4) - \tfrac{5}{72}(J_x^6 + J_y^6), \tag{28}$$

which tolerates white noise if $p_{\mathrm{noise}} < 0.1391$. Straightforward calculation shows that $\mathcal{W}_{\mathrm{D}(6,3)}^{(P2)} - 2.5\mathcal{W}^{(P)} \geqslant 0$. Based on (17), $(0.6 - \langle\mathcal{W}_{\mathrm{D}(6,3)}^{(P2)}\rangle/2.5)$ bounds the fidelity from below.

3.1.2. Three-setting witness. Similarly we can look for the optimal witness for the three-setting case. The result is

$$\mathcal{W}_{\mathrm{D}(6,3)}^{(P3)} := 1.5 \cdot \mathbb{1} - \tfrac{1}{45}(J_x^2 + J_y^2) + \tfrac{1}{36}(J_x^4 + J_y^4) - \tfrac{1}{180}(J_x^6 + J_y^6) + \tfrac{1007}{360}J_z^2 - \tfrac{31}{36}J_z^4 + \tfrac{23}{360}J_z^6. \tag{29}$$

White noise is tolerated if $p_{\mathrm{noise}} < 0.2735$. It is easy to check that $\mathcal{W}$ is a witness as $\mathcal{W}_{\mathrm{D}(6,3)}^{(P3)} - 2.5\mathcal{W}^{(P)} \geqslant 0$.

Based on (17), the expectation value of this witness can be used to bound the fidelity as $F \geqslant 0.6 - \langle\mathcal{W}_{\mathrm{D}(6,3)}^{(P3)}\rangle/2.5 =: F'$. Here we will demonstrate how well the fidelity estimation works for our witness for noisy states. We consider first white noise, then non-white noise of the form

$$\varrho_{\mathrm{noisy}}^{(\mathrm{NW})} := p_{D63}|D_6^{(3)}\rangle\langle D_6^{(3)}| + \frac{1 - p_{D63}}{2}\left(|D_6^{(2)}\rangle\langle D_6^{(2)}| + |D_6^{(4)}\rangle\langle D_6^{(4)}|\right), \tag{30}$$

with $p_{D63} = 4/7$, which is one of the relevant types of noise for the experiment of [32]. Note that the noise contains the original state $|D_6^{(3)}\rangle\langle D_6^{(3)}|$. The results are shown in figure 1. For the non-white noise (30), the fidelity estimate based on the witness yields a very good estimate.

Note that it is also possible to design a witness for the largest possible tolerance to the noise in (30). Due to the special form of the noise, the fidelity estimate turns out to be equal to the fidelity. This is remarkable: the fidelity can be obtained exactly with only three local measurements.

3.1.3. Measuring the projector-based witness. For measuring the projector-based witness (7) for $N = 6$, one has to decompose the projector in an efficient way. The straightforward decomposition into the weighted sum of products of Pauli spin matrices leads to a scheme that needs 183 settings, since for all local operators all the permutations have to be measured. The number of settings needed can be dramatically decreased if one is looking for a decomposition of the form (9). Observation 1 makes it possible to decompose the projector in this way such that only 25 settings are needed. We could further decrease the number of settings needed and found the following decomposition:

$$64|D_6^{(3)}\rangle\langle D_6^{(3)}| = -0.6[\mathbb{1}] + 0.3[x \pm \mathbb{1}] - 0.6[x] + 0.3[y \pm \mathbb{1}] - 0.6[y] + 0.2[z \pm \mathbb{1}] - 0.2[z]$$

$$+ 0.2\mathrm{Mermin}_{0,z} + 0.05[x \pm y \pm \mathbb{1}] - 0.05[x \pm z \pm \mathbb{1}] - 0.05[y \pm z \pm \mathbb{1}]$$

$$- 0.05[x \pm y \pm z] + 0.2[x \pm z] + 0.2[y \pm z] + 0.1[x \pm y]$$

$$+ 0.6\mathrm{Mermin}_{x,z} + 0.6\mathrm{Mermin}_{y,z}. \tag{31}$$
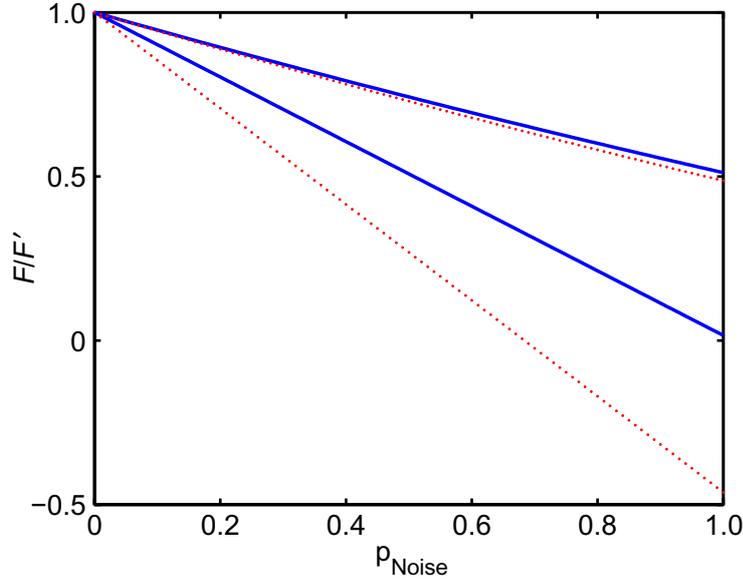
Figure 1. The fidelity F versus noise (solid) and the fidelity estimate F' versus noise (dotted), for the white noise (bottom two curves) and for the non-white noise (30) (top two curves). For the fidelity estimate, the three-setting witness (29) was used.

Here we use the notation $[x + y] = (\sigma_x + \sigma_y)^{\otimes 6}$, $[x + y + \mathbb{1}] = (\sigma_x + \sigma_y + \mathbb{1})^{\otimes 6}$, etc. The $\pm$ sign denotes a summation over the two signs, i.e., $[x \pm y] = [x + y] + [x - y]$. The Mermin operators are defined as

$$\text{Mermin}_{a,b} := \sum_{k \, \text{even}} (-1)^{k/2} \sum_k \mathcal{P}_k (\otimes_{i=1}^k \sigma_a \otimes_{i=k+1}^N \sigma_b), \qquad (32)$$

where $\sigma_0 = \mathbb{1}$. That is, it is the sum of terms with even number of σ_a's and σ_b's, with the sign of the terms depending on the number of σ_a's. The expectation value of the operators $\text{Mermin}_{a,b}$ can be measured based on the decomposition [23]

$$\text{Mermin}_{a,b} = \frac{2^{N-1}}{N} \sum_{k=1}^N (-1)^k \left[\cos\left(\frac{k\pi}{N}\right) a + \sin\left(\frac{k\pi}{N}\right) b \right]^{\otimes N}. \qquad (33)$$

Hence, $\text{Mermin}_{x,z}$ and $\text{Mermin}_{y,z}$ can be measured with six settings. $\text{Mermin}_{0,z}$, on the other hand, needs only the measurement of the $\{\sigma_z, \sigma_z, \sigma_z, \sigma_z, \sigma_z, \sigma_z\}$ setting. Knowing that $[A]$, $[A + \mathbb{1}]$ and $[A - \mathbb{1}]$ can be measured with a single setting $\{A, A, A, \ldots, A\}$, we find that 21 measurement settings are needed to measure $|D_6^{(3)}\rangle\langle D_6^{(3)}|$: $x, y, z, x \pm y, x \pm z, y \pm z, \sqrt{3}x \pm z, \sqrt{3}z \pm x, \sqrt{3}y \pm z, \sqrt{3}z \pm y$, and $x \pm y \pm z$. The settings are also shown in figure 2(a)[11].

3.2. Witness independent from the projector witness

So far we constructed witnesses that detected fewer states than the projector-based witness, in return, they were easier to measure. When proving that they were witnesses, we used the

---

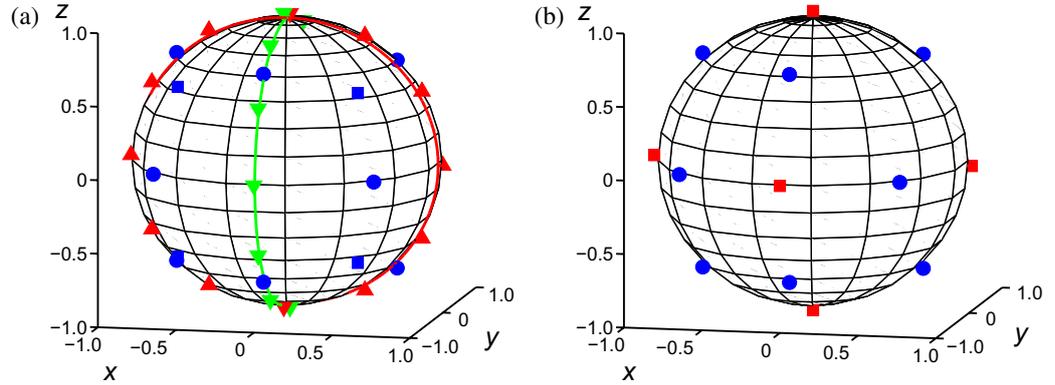[11] Note that [33] presents another decomposition that needs also 21 settings.

Figure 2. (a) The measurement settings needed to measure the projector to the six-qubit symmetric Dicke state with three excitations based on the decomposition (31). A point at (x, y, z) indicates measuring $x\sigma_x + y\sigma_y + z\sigma_z$ on all qubits. (▲) Settings for $\text{Mermin}_{x,z}$, (▼) settings for $\text{Mermin}_{y,z}$, (■) $\sigma_x \pm \sigma_y \pm \sigma_z$, and (●) rest of the settings. (b) Settings for the four-qubit Dicke state with two excitations based on (37). (■) $\pm\sigma_x, \pm\sigma_y, \pm\sigma_z$, and (●) $\sigma_x \pm \sigma_y, \sigma_x \pm \sigma_z$, and $\sigma_y \pm \sigma_z$.

Table 1. The list of entanglement witnesses presented in this paper, together with the number of measurement settings needed to measure them and their robustness to white noise. Top four lines: six-qubit witnesses. Bottom five lines: four- and five-qubit witnesses.

Witness	Number of settings	Noise tolerance
$\mathcal{W}_{\text{D}(6,3)}^{(P)}$	21	0.4063
$\mathcal{W}_{\text{D}(6,3)}^{(P3)}$	3	0.2735
$\mathcal{W}_{\text{D}(6,3)}^{(P2)}$	2	0.1391
$\mathcal{W}_{\text{D}(6,3)}^{(I2)}$	2	0.1091
$\mathcal{W}_{\text{D}(5,2)}^{(I2)}$	2	0.1046
$\mathcal{W}_{\text{D}(4,1)}^{(P)}$	7	0.2667
$\mathcal{W}_{\text{D}(4,1)}^{(I2)}(q = 1.47)$	3	0.1476
$\mathcal{W}_{\text{D}(4,2)}^{(P)}$	9	0.3556
$\mathcal{W}_{\text{D}(4,2)}^{(P3)}$	2	0.2759

simple relation (2). Following the example of [37], we now look for a two-setting witness of the form (26) for $N = 6$ that is independent from the projector witness. For determining c_{D6}, we need to compute the maximum of $J_x^2 + J_y^2$ for biseparable states for all the possible bipartitions. As we have discussed in section 2.3, instead of looking for the maximum for states that are

separable with respect to a certain bipartition, we can also look for the maximum for PPT states (see appendix A). We obtain

$$c_{D6} := 11.0179. \tag{34}$$

$\mathcal{W}_{\text{D}(6,3)}^{(I2)}$ detects genuine multipartite entanglement if for white noise $p_{\text{noise}} < 0.1091$. Simple numerical optimization leads to the same value for the maximum for biseparable states[12]. Hence we find that our witness is optimal. Finally, the list of witnesses presented in this section are shown in the top part of table 1.

4. Witnesses for states derived from $|D_6^{(3)}\rangle$ via projections

By projective measurements of one or two of the qubits we can obtain several states that are inequivalent under stochastic local operations and classical communication (SLOCC). Surprisingly, these states still possess genuine multipartite entanglement [32, 44]. Next, we discuss how to detect the entanglement of these states.

4.1. Witnesses for the superposition of five-qubit Dicke states

After measuring one of the qubits in some basis and post-selecting for one of the two outcomes, one can obtain states of the form

$$\varrho_{\text{D5}} := c_1|D_5^{(2)}\rangle + c_2|D_5^{(3)}\rangle, \tag{35}$$

where $|c_1|^2 + |c_2|^2 = 1$. For such states, the expectation value of $J_x^2 + J_y^2$ is maximal, thus a witness of the form (26) for $N = 5$ is used to detect their entanglement. Both semidefinite programing and simple numerical optimization leads to $c_{D5} := 7.8723$. Naturally, $\langle \mathcal{W}_{\text{D5}}^{(I2)} \rangle$ is minimal not only for states of the form (35), but for any mixture of such states.

4.2. Witness for the four-qubit W state

Now we will construct witnesses for a four-qubit W state, which is obtained from $|D_6^{(3)}\rangle$ if two qubits are measured in the σ_z basis, and the measurement result is +1 in both cases. We consider a witness of the form (27) for $N = 4$ and $m = 1$. We try several values for q and determine c_q for the witness $\mathcal{W}_{\text{D}(4,1)}(q)$ as a function of q using semidefinite programming. For each witness we also compute the noise tolerance. The results of these computations can be seen in figure 3. It turns out, that the best witness is obtained for $q = 1.47$ and $c_q = 4.1234$. It tolerates white noise if $p_{\text{noise}} < 0.1476$.

4.3. Three-setting witness for the four-qubit Dicke state

A $|D_4^{(2)}\rangle$ state can also be obtained from $|D_6^{(3)}\rangle$, namely if the measurement outcomes are +1 and −1 for two consecutive σ_z measurements. For that case, we look for a three-setting witness, based on the projector witness. For white noise, the result is

$$\mathcal{W}_{\text{D}(4,2)}^{(\text{P3})} := 2 \cdot \mathbb{1} + \tfrac{1}{6}(J_x^2 + J_y^2 - J_x^4 - J_y^4) + \tfrac{31}{12}J_z^2 - \tfrac{7}{12}J_z^4. \tag{36}$$

---

[12] We used the `maxbisep` routine of the QUBIT4MATLAB V3.0 package [34] with parameters for accuracy [30 000, 100 000, 0.0005]. See also appendix B.
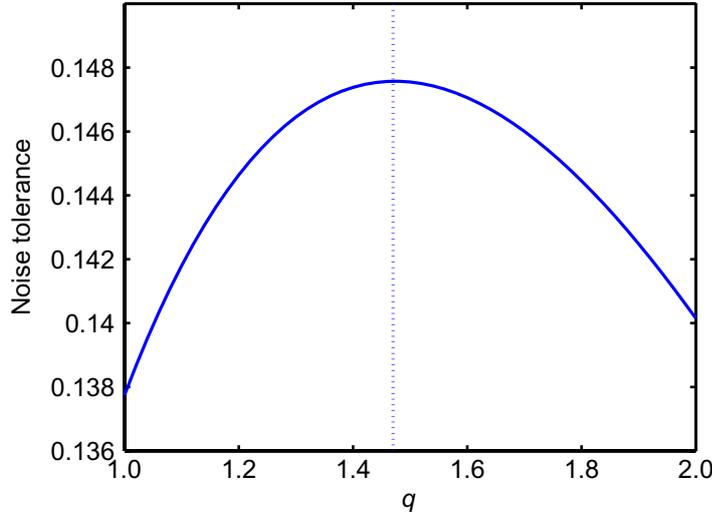
Figure 3. The noise tolerance of the witness $\mathcal{W}_{\mathrm{D}(1,4)}^{(I3)}$ given in (27) as a function of the parameter q. The maximum is in the vicinity of $q = 1.47$.

The witness tolerates white noise if $p_{\mathrm{noise}} < 0.2759$. It is easy to check that $\mathcal{W}$ is a witness: one has to notice that $\mathcal{W}_{\mathrm{D}(4,2)}^{(P3)} - 3\mathcal{W}_{\mathrm{D}(4,2)}^{(P)} \geqslant 0$, were $\mathcal{W}_{\mathrm{D}(4,2)}^{(P)}$ is defined in (7). Thus, the fidelity can be estimated from the measurement of the witness as $F \geqslant 2/9 - \langle \mathcal{W}_{\mathrm{D}(4,2)}^{(P3)} \rangle / 3$.

4.4. Measuring the projector witness for the four-qubit Dicke state

We can also measure the projector witness $\mathcal{W}_{\mathrm{D}(4,2)}^{(P)} = \frac{2}{3}\mathbb{1} - |D_4^{(2)}\rangle\langle D_4^{(2)}|$. The method in observation 1 gives the following decomposition for the projector

$$16|D_4^{(2)}\rangle\langle D_4^{(2)}| = \frac{2}{3}([x] + [x \pm \mathbb{1}] + [y] + [y \pm \mathbb{1}]) + \frac{1}{3}(8[z] - [z \pm \mathbb{1}] - [x \pm z] - [y \pm z])$$

$$+ \frac{1}{6}[x \pm y]. \tag{37}$$

The nine measurement settings are x, y, z, $x \pm y$, $x \pm z$ and $y \pm z$, shown also in figure 2(b). The list of witnesses presented in this section are given in the bottom part of table 1.

5. Conclusions

In summary, we presented general methods for constructing entanglement witnesses for detecting genuine multipartite entanglement in experiments. In particular, we considered projector-based witnesses and found efficient decompositions for them. Then, we constructed two- and three-setting witnesses for symmetric Dicke states that were based on the projector witness, as well as independent of the projector witness. We applied our methods to design witnesses for the recent experiment observing a six-qubit symmetric Dicke state with three excitations [32]. Our methods can be generalized for future experiments. As a first step, in appendix C we list some entanglement witnesses for systems with 5–10 qubits. Moreover, recent results on the symmetric tensor rank problem suggest that decompositions more efficient than the one in observation 1 are possible, however, they involve complex algorithms [45]. Thus, it would be interesting to look for better upper bounds for the number of settings used for symmetric operators.

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Acknowledgments

Appendix A. Semidefinite programming used for obtaining witnesses

Here we summarize two optimization problems that are useful for designing entanglement witnesses and can be solved with semidefinite programming. Both tasks are related to designing witnesses that are easy to measure.

1. Semi-definite programming can be used to find the witness $\mathcal{W}$ with the largest noise tolerance as explained in the beginning of section 3.1.1. The corresponding task can be formulated as

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_k c_k \text{Tr}(B_k \varrho_{\text{noise}}), \\
\textbf{subject to} \quad & \sum_k c_k \text{Tr}(B_k \varrho) = -1, \\
& \sum_k c_k B_k - \alpha \mathcal{W}^{(P)} \geqslant 0, \\
& \alpha > 0.
\end{aligned}
\tag{A.1}
$$

Here ϱ is the state around which we detect entanglement. ϱ_{noise} is the noise, not necessarily white. The optimization is over α and the c_k's.

2. Semi-definite programming can be used to look for the maximum for PPT bipartite states. This gives an upper bound on the maximum for biseparable states. In many cases, the two coincide. The corresponding task can be formulated as a standard semidefinite program as

$$
\begin{aligned}
\textbf{minimize} \quad & -\text{Tr}(M\varrho), \\
\textbf{subject to} \quad & \varrho \geqslant 0, \\
& \text{Tr}(\varrho) = 1, \\
& \varrho^{T_A} \geqslant 0.
\end{aligned}
\tag{A.2}
$$

Here T_A means partial transpose according to some groups of the qubits.

Appendix B. List of MATLAB subroutines

We summarize some of the MATLAB routines of the QUBIT4MATLAB 3.0 package that can be used for the calculations necessary for designing entanglement witnesses. A full list of the commands is given in [34].

The command `decompose` can be used to obtain a decomposition of a Hermitian operator into the sum of products of Pauli spin matrices. Moreover, `maxsep` and `maxsymsep` can be

used for getting the maximum for separable multi-qudit states and symmetric product states for a Hermitian operator, respectively. The command `maxbisep` gives the maximum for states that are biseparable with respect to some partitioning of the qubits. The command `maxb` gives the maximum for all possible bipartitions. All these commands look for the maximum with a simple optimization algorithm that is not guaranteed to find the global maximum, nevertheless, it typically does find it. `overlapb` gives the maximum overlap of a state $|\Psi\rangle$ and biseparable states. It can be used to construct entanglement witnesses of the type (6).

For semidefinite programming, we used SeDuMi [46] and YALMIP [47]. Two subroutines based on them are now in QUBIT4MATLAB 4.0[13]. The command `optwitness` looks for the best witness that can be composed linearly from a set of operators, while `maxppt` determines the maximum of an operator expectation value for states with a PPT for some bipartitioning of the qubits.

Appendix C. Witnesses for systems with 5–10 qubits

A three-setting witness based on the projector witness for the state $|D_8^{(4)}\rangle$ is given by

$$\mathcal{W}_{\mathrm{D}(8,4)}^{(P3)} := 1.3652 \cdot \mathbb{1} + \sum_{l=x,y,z} \sum_{n=1}^{4} c_{ln} J_l^{2n}, \tag{C.1}$$

with

$$\{c_{ln}\} = \begin{pmatrix} 0.0038612 & -0.0052555 & 0.0015016 & -0.00010726 \\ 0.0038612 & -0.0052555 & 0.0015016 & -0.000107266 \\ 3.124 & -1.07699 & 0.11916 & -0.0038992 \end{pmatrix}. \tag{C.2}$$

The noise tolerance for white noise is $p_{\mathrm{noise}} < 0.2578$. For larger N, we can use the ansatz

$$\mathcal{W}_{\mathrm{D}(N,N/2)}^{(P3)} := c_1 \cdot \mathbb{1} + c_{xy}\{(\sigma_x + \mathbb{1})^{\otimes N} + (\sigma_x - \mathbb{1})^{\otimes N} + (\sigma_y + \mathbb{1})^{\otimes N} + (\sigma_y - \mathbb{1})^{\otimes N}\} + \sum_{n=1}^{N/2} c_{zn} J_z^{2n}. \tag{C.3}$$

For $N = 10$, the optimal coefficients are $c_1 = 1.3115$, $c_{xy} = -0.0023069$, and $c_z = \{3.4681, -1.2624, 0.16494, -0.0084574, 0.000146551\}$. White noise is tolerated if $p_{\mathrm{noise}} < 0.2404$. The large noise-tolerance for the $N = 10$ case suggests that a robust three-setting witness for $|D_N^{(N/2)}\rangle$ might be constructed even for large N.

A three-setting witness independent of the projector witness for the N-qubit W state is given by (27) for $m = 1$. For $N = 5$, we have $c_5 = 5.6242$, $q_5 = 2.22$, and the witness tolerates white noise if $p_{\mathrm{noise}} < 0.0744$. For $N = 6$, we have $c_6 = 7.1095$, $q_6 = 3.13$, and noise is tolerated if $p_{\mathrm{noise}} < 0.0401$.

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